

# Respecting Priorities when Assigning Students to Schools\*

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## Abstract

We consider the problem of assigning students to schools on the basis of priorities. Students are allowed to have equal priority at a school. We characterize the efficient rules which weakly/strongly respect students' priorities. When priority orderings are not strict, it is not possible to simply break ties in a fixed manner. All possibilities of resolving the indifferences need to be considered for an efficient assignment of students to schools. If in addition we insist on incentive-compatibility, then neither the deferred acceptance algorithm nor the top trading cycle algorithm successfully solve the problem of efficiently assigning the students to schools whereas a modified version of the deferred acceptance algorithm might. In this version tie breaking depends on students' preferences.

*Keywords:* School Choice, Equal Priority, Tie Breaking.

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# 1 Introduction

Until the 1990s children were assigned to public schools in the district where their parents live in. Rich families could choose the district where their children will go to school by moving to that district. Poor families did not have this choice. Since “better” public schools are often located in richer neighborhoods, children from poor families did not have access to those schools. This inequality of opportunity between rich and poor families triggered U.S. cities to establish centralized interdistrict school choice programs. Such a program offers students (or children) the possibility to be admitted at schools which are not located in the district in which they live. It assigns students to schools (or universities) on the basis of priorities and students’ (or parents’) preferences. A school’s priority ordering represents the rights students have to be admitted at that school and is based on observed characteristics of students or an entry exam. For example, the city of Boston determines the priority ordering of a school as follows (Abdulkadiroğlu and Sönmez, 2003): first priority is given to students who live within walking distance from the school and already have a sibling attending the school; second priority is given to students who have a sibling attending the school; third priority is given to students who live within walking distance from the school; and fourth priority is given to all other students. Any two students with the same characteristics have equal priority. Priorities are fixed because they are obtained through an objective test or by law. The only criteria for efficiency and strategic manipulation of rules are students’ preferences. Several papers<sup>1</sup> consider a special case of this problem where schools’ priority orderings are strict. In real applications, however, priority orderings are rarely strict. These authors assumed that when a priority ordering is not strict, one can simply a priori (before the revelation of students’ preferences) break ties randomly to obtain a strict priority ordering. We will show that this approach causes several difficulties. When priority orderings are

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<sup>1</sup>See Balinski and Sönmez (1999), Abdulkadiroğlu and Sönmez (2003), Ergin (2002), Kesten (2006), Ergin and Sönmez (2006), and Abdulkadiroğlu and Ehlers (2007).

not strict, we need to consider **all** possible resolutions of indifferences in a priority ordering. Students having equal priorities at a school are then ordered in both ways.

The following is an intuitively appealing principle for rules to follow students' priorities: if student  $i$  has higher priority than student  $j$  at school  $a$ , then it should not be the case that student  $i$  envies student  $j$  because student  $j$  was admitted by school  $a$ . We say that an assignment of the students to the schools *weakly respects* the priority orderings of the schools if such a violation does not occur. Likewise a rule weakly respects the priority orderings of the schools if for any students' preferences the assignment chosen by the rule weakly respects the priority orderings of the schools. For strict priority orderings Balinski and Sönmez (1999) and Ergin (2002) use the deferred acceptance (DA) algorithm with students proposing<sup>2</sup> to assign students to schools. The DA-algorithm weakly respects the strict priority orderings of the schools and is Pareto superior to any other rule which weakly respects the strict priority orderings of the schools. Unfortunately the DA-algorithm does not break ties and is not defined for weak priority orderings.

When students are allowed to have equal priorities, there is another natural principle for rules to follow students' priorities: if student  $i$  has at least the same priority as student  $j$  at school  $a$ , then it should not be the case that student  $i$  envies student  $j$  because student  $j$  was admitted by school  $a$ . This principle is especially important when ties are broken randomly: a tie may be broken either way and each of the students may be ranked higher than the other. A priori both tied students have the same priority and when breaking the tie either way, in the strict resolution each of the tied students may have higher priority than the other. Similarly as above we say that a rule *strongly respects* the priority orderings of the schools if such a violation does not occur for any students' preferences. Our first result shows that if an efficient rule strongly respects the priority orderings of the schools, then the DA-algorithm yields for all resolutions of indifferences the same assignment. Then each school's priority

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<sup>2</sup>This algorithm is due to Gale and Shapley (1962).

ordering is “essentially” strict and we may apply the DA-algorithm to any fixed tie breaking. However, these rules only strongly respect the priorities since they all yield the same rule and tie breaking is not necessary. Our second result shows that if an efficient rule weakly respects the priority orderings of the schools, then for each profile there exists a resolution of indifferences in schools’ priority orderings such that the assignment chosen by the rule is obtained from applying the DA-algorithm to that resolution. This result does not say that we can break ties exogenously and apply to each profile the DA-algorithm with the fixed tie breaking.

Most rules which are used in real life have the flaws that they are inefficient and manipulable. Abdulkadiroğlu and Sönmez (2003) suggest two mechanisms for solving the school choice problem with strict priority orderings. The first mechanism is the DA-algorithm and the second mechanism is a generalized version of Gale’s top trading cycle algorithm, called “hierarchical exchange rule” (Pápai, 2000). The merits and disadvantages of these mechanisms are the following: (1) The DA-algorithm weakly respects the priority orderings of the schools but may be inefficient. (2) A hierarchical exchange rule is efficient but may not weakly respect the priority orderings of the schools. We show that neither approach successfully solves the school choice problem when equal priorities are allowed. There are school choice problems such that (i) no DA-algorithm (with fixed tie breaking) is efficient, (ii) no hierarchical exchange rule weakly respects the priority orderings of the schools, and (iii) there exists an efficient and group strategy-proof rule which weakly respects the priority orderings of the schools. The rule we propose is a modified version of the DA-algorithm in which ties are broken endogenously, i.e. resolution of indifferences in a priority ordering depends on students’ preferences. This rule takes into account all possibilities of breaking ties.

The paper is organized as follows. Section 2 defines the model and states several well-known facts for strict priority structures. Section 3 introduces the Gale-Shapley correspondence and shows via an example the difficulties of assigning students to schools under weak priority structures. Section 4 generalizes the main result of Ergin

(2002) to rules strongly respecting a weak priority structure. Section 5 shows that a modified version of the DA-algorithm with endogenous tie-breaking may outperform both the DA-algorithm with fixed tie-breaking and hierarchical exchange with fixed tie-breaking. It also shows that the main result of Ergin (2002) cannot be generalized to rules weakly respecting a weak priority structure. Section 6 concludes.

## 2 The Model

Let  $N$  denote the set of students and  $A$  the set of schools. Each school  $a \in A$  has a certain number of available positions, denoted by  $q_a \in \mathbb{N}$ . Sometimes we refer to  $q_a$  as the *quota* of school  $a$ . Let  $q \equiv (q_a)_{a \in A}$  denote the list of quotas. Each student is equipped with a preference relation  $R_i$  over  $A \cup \{i\}$  where  $i$  stands for not being assigned to any school. Let  $P_i$  denote the strict preference relation associated with  $R_i$ . We assume that each student's preference relation  $R_i$  is complete, transitive, and antisymmetric, i.e. for all  $a, b \in A \cup \{i\}$  such that  $a \neq b$ ,  $aP_ib$  or  $bP_ia$ . A (*preference*) *profile*  $R$  is a list  $(R_i)_{i \in N}$ . Given profile  $R$  and  $M \subseteq N$ , let  $R_M$  denote the restriction of  $R$  to  $M$ . Let  $\mathcal{R}$  denote the set of all profiles. In an assignment, each student is assigned to a school or to no school, and the number of students who are assigned to a school does not exceed its quota. Formally, an *assignment* is a function  $\mu : N \rightarrow A \cup N$  satisfying (i) for all  $i \in N$ ,  $\mu(i) \in A \cup \{i\}$ , and (ii) for all  $a \in A$ ,  $|\mu^{-1}(a)| \leq q_a$ . Each school  $a$  has a (complete and transitive) *priority ordering*  $\succeq_a$  over the set of students. A *priority structure*  $\succeq$  is a list  $(\succeq_a)_{a \in A}$ . For all  $a \in A$  and all  $i \in N$ , let  $U_a(i) \equiv \{j \in N \mid j \succ_a i\}$  denote the strict upper contour set of  $i$  at  $\succeq_a$  and  $W_a(i) \equiv \{j \in N \mid j \succeq_a i\}$  the weak upper contour set of  $i$  at  $\succeq_a$ . For all  $a \in A$  and all  $N' \subseteq N$ , let  $\succeq_a|_{N'}$  denote the restriction of  $\succeq_a$  to the set  $N'$ .

An *assignment problem (with a priority structure) (or school choice problem)* is a quintuple  $(N, A, (q_a)_{a \in A}, R, \succeq)$ . Because everything but  $R$  remains fixed, an assignment problem is simply a profile  $R$ . A *rule* is a function  $f$  mapping profiles to assignments. Given a profile  $R$  and student  $i$ ,  $f_i(R)$  denotes  $i$ 's assignment in  $f(R)$ .

A rule  $f$  is *efficient* if for any profile  $R$  there does not exist an assignment  $\mu$  which is Pareto superior to  $f(R)$ , i.e. there is no  $\mu$  such that for all  $i \in N$ ,  $\mu(i)R_i f_i(R)$ , and for some  $j \in N$ ,  $\mu(j)P_j f_j(R)$ . A rule  $f$  is *strategy-proof* if no student can gain by misrepresentation, i.e. for all  $i \in N$ , all  $R \in \mathcal{R}$ , and all  $R'_i$ ,  $f_i(R)R_i f_i(R_{N \setminus \{i\}}, R'_i)$ . A rule  $f$  is *group strategy-proof* if no group of students can gain by joint misrepresentation, i.e. for all  $M \subseteq N$ , all  $R \in \mathcal{R}$ , and all  $R'_M = (R'_i)_{i \in M}$ , if for some  $i \in M$ ,  $f_i(R_{N \setminus M}, R'_M)P_i f_i(R)$ , then for some  $j \in M$ ,  $f_j(R)P_j f_j(R_{N \setminus M}, R'_M)$ . A rule  $f$  has a *consistent extension* if its choices involving different school choice problems are coherent, i.e. for all  $M \subseteq N$ , all  $(R_M, R_{N \setminus M}) \in \mathcal{R}$ , and all  $R'_{N \setminus M} = (R'_i)_{i \in N \setminus M}$ , if for all  $a \in A$ ,  $|\{i \in N \setminus M | f_i(R_M, R_{N \setminus M}) = a\}| = |\{i \in N \setminus M | f_i(R_M, R'_{N \setminus M}) = a\}|$ , then for all  $i \in M$ ,  $f_i(R_M, R_{N \setminus M}) = f_i(R_M, R'_{N \setminus M})$ . A *correspondence* is a function mapping profiles to non-empty sets of assignments. We say that a rule  $f$  is a *selection* from correspondence  $F$  if for any profile  $R$ ,  $f(R)$  belongs to  $F(R)$ .

Since we allow students to have equal priorities at a school, there are two principles for rules to follow students' priorities. These are the analogous definitions to strong and weak core, respectively, and weak and strong blocking, respectively.

The first principle says that if a student is assigned to a certain school, then there should be no student who both envies him and has at least the same priority to be admitted at that school.

**Definition 1** *Let  $\succeq$  be a priority structure and  $R \in \mathcal{R}$ . The assignment  $\mu$  weakly violates the priority of  $i$  for  $a$  at  $\mu$  if for some  $j \in N$  we have  $\mu(j) = a$ ,  $aP_i \mu(i)$ , and  $i \succeq_a j$ . A rule  $f$  strongly respects  $\succeq$  if for all profiles  $R \in \mathcal{R}$ ,  $f(R)$  does not weakly violate any priorities.*

The second principle says that if a student is assigned to a certain school, then there should be no student who both envies him and has higher priority to be admitted at that school.

**Definition 2** *Let  $\succeq$  be a priority structure and  $R \in \mathcal{R}$ . The assignment  $\mu$  strongly violates the priority of  $i$  for  $a$  at  $\mu$  if for some  $j \in N$ , we have  $\mu(j) = a$ ,  $aP_i \mu(i)$ , and*

$i \succ_a j$ . A rule  $f$  weakly respects  $\succeq$  if for all profiles  $R \in \mathcal{R}$ ,  $f(R)$  does not strongly violate any priorities.

Definitions 1 and 2 are equivalent when the priority ordering of each school is strict. Specifically, the first principle applies to school choice problems where ties are broken randomly: a priori two tied students have the same priority and when breaking the tie in either way, in the strict resolution each of the tied students may have higher priority than the other.

The priority structure  $\succeq$  represents *fixed* “preferences” of the schools. A school’s priority ordering was obtained through test scores of an entry exam or from observable characteristics of a student (such as the district where the student lives and whether siblings of the student already attend the school). Grades of students on exams often fall into a small number of discrete bins. If students are prioritized by grades, this can lead to large indifference classes. Priority orderings are obtained through an objective test and are thus irrelevant from a welfare perspective. Students’ preferences are the only criteria for efficiency. Similarly students’ preferences evoke strategic considerations whereas priority orderings do not.

Several authors have studied the special case where schools’ priority orderings are strict. Given a strict priority structure  $\succeq$  and a profile  $R$ , Balinski and Sönmez (1999) and Ergin (2002) applied the DA-algorithm with students proposing to  $(\succeq, R)$ :<sup>3</sup>

At the first step, every student applies to his favorite acceptable school. For each school  $a$ ,  $q_a$  applicants who have the highest priority for  $a$  (all applicants if there are fewer than  $q_a$ ) are placed on the waiting list of  $a$ , and the others are rejected.

At the  $r$ th step, those applicants who were rejected at step  $r - 1$  apply to their next best acceptable schools. For each school  $a$ , the highest priority  $q_a$  students among the new applicants and those in the waiting list are placed on the new waiting list and the rest are rejected.

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<sup>3</sup>A more detailed discussion of the remaining part of this section can be found in Ergin (2002).

The DA-algorithm terminates when every student is either on a waiting list or has been rejected by every school that is acceptable to him. After this procedure ends, schools admit the students on their waiting list, which yields the Gale-Shapley assignment for the problem  $(\succeq, R)$ . We denote this assignment by  $f^\succeq(R)$ . It follows from “pairwise stability”<sup>4</sup> of the DA-algorithm that the Gale-Shapley assignment does not strongly violate any priorities. It even turns out that this assignment is Pareto superior to any other assignment which does not strongly violate any priorities.

**Proposition 1 (Balinski and Sönmez, 1999)** *For any **strict** priority structure  $\succeq$  and any profile  $R$ ,*

- (a) *the assignment  $f^\succeq(R)$  weakly respects  $\succeq$ , and*
- (b) *the assignment  $f^\succeq(R)$  is Pareto superior to any other assignment which weakly respects  $\succeq$ .*

When the priority structure  $\succeq$  is strict, we denote by  $f^\succeq$  the Gale-Shapley rule taking as input the fixed priority orderings of the schools. Therefore, by Proposition 1, if  $\succeq$  is strict, then  $f^\succeq$  is a rule which weakly respects  $\succeq$ .

Ergin (2002) introduced the following definition of a cycle.

**Definition 3 (Ergin-cycle)** *Let  $\succeq$  be a priority structure and  $q$  be a list of quotas. Then  $\succeq$  contains an Ergin-cycle if there are  $a, b \in A$  and  $i, j, k \in N$  such that the following conditions are satisfied:*

- (EC) *Ergin-cycle condition:  $i \succ_a j \succ_a k$  and  $k \succ_b i$ .*
- (ES) *Ergin-scarcity condition: There exist (possibly empty) disjoint sets  $N_a, N_b \subseteq N \setminus \{i, j, k\}$  such that  $N_a \subseteq U_a(j)$ ,  $N_b \subseteq U_b(i)$ ,  $|N_a| = q_a - 1$ , and  $N_b = q_b - 1$ .*

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<sup>4</sup>An assignment  $\mu$  is *pairwise stable* under  $(\succeq, R)$  if there does not exist a pair of a school  $a$  and a student  $i$  such that  $aP_i\mu(i)$  and for some  $j \in \mu^{-1}(a)$ ,  $i \succ_a j$ .

When  $\succeq$  is strict, Ergin (2002) shows that the non-existence of an Ergin-cycle is a necessary and sufficient condition for  $f^\succeq$  to be efficient or group strategy-proof or to have a consistent extension.<sup>5</sup>

**Theorem 1 (Ergin, 2002)** *Let  $\succeq$  be a **strict** priority structure and  $q$  be a list of quotas. The following are equivalent:*

- (i)  $f^\succeq$  is efficient.
- (ii)  $f^\succeq$  is group strategy-proof.
- (iii)  $f^\succeq$  has a consistent extension.
- (iv)  $\succeq$  does not contain any Ergin-cycle.

### 3 The Gale-Shapley Correspondence

When a priority structure is not strict, the DA-algorithm is not well-defined: when two students apply to a school which assigns equal priority to them, the school does not know which student to reject. If we would try to use the DA-algorithm, we would need to transform the (weak) priority structure into a strict priority structure. A natural candidate for such a strict priority structure would be one in which we resolve indifferences and preserve strict preferences.<sup>6</sup> However, as we will show, it is not sufficient just to apply a fixed resolution of indifferences in the given priority structure. Efficient rules need to incorporate **all** possibilities of breaking ties. For any assignment problem we will choose all Gale-Shapley assignments which are obtained from applying the DA-algorithm to a resolution of indifferences.

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<sup>5</sup>In the model of assigning students to schools without any given priority structure, Ehlers and Klaus (2006) characterize the DA-algorithms where  $\succeq$  is strict and does not contain any Ergin-cycle. Kojima and Manea (2006) characterize the DA-algorithms where  $\succeq$  is strict.

<sup>6</sup>This is also suggested by Roth and Rothblum (1999, p.25).

Formally, given two priority structures  $\succeq$  and  $\succeq'$ , we call  $\succeq'$  a *strict resolution* of  $\succeq$  if for all  $a \in A$ , (i)  $\succeq'_a$  is strict and (ii) the strict preference of  $\succeq_a$  is preserved under  $\succeq'_a$ , i.e. for all  $i, j \in N$ , if  $i \succ_a j$ , then  $i \succ'_a j$ . For all profiles  $R \in \mathcal{R}$ , let the *Gale-Shapley correspondence (with priority structure  $\succeq$ )*  $F^\succeq$  choose all assignments obtained from applying the DA-algorithm to a strict resolution of  $\succeq$  and  $R$ , i.e.

$$F^\succeq(R) \equiv \cup_{\succeq' \text{ is a strict resolution of } \succeq} f^{\succeq'}(R).$$

The following example illustrates some of the difficulties of assigning students to schools under weak priorities and relates those difficulties to the Gale-Shapley correspondence.

**Example 1 (House Allocation with Existing Tenants)** Let  $N = \{1, 2, 3, 4\}$ ,  $A = \{a, b, c, d\}$ , and  $q_a = q_b = q_c = q_d = 1$ . Let  $R$  be any profile and  $\succeq$  be the priority structure such that

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	2	3	1, 2, 3, 4
2, 3, 4	1, 3, 4	1, 2, 4	

Here the existing tenants are 1, 2, and 3 (where 1 lives in  $a$ , 2 lives in  $b$ , and 3 in  $c$ ). House  $d$  is vacant and all the students have the same right on it.

The first difficulty is that a priori all students have equal priority for house  $d$  and we cannot assign house  $d$  without weakly violating any priorities. Indeed, depending on how we break ties, different students are assigned house  $d$  under the DA-algorithm and the Gale-Shapley correspondence is multi-valued. Later we will see that single-valuedness of the Gale-Shapley correspondence is necessary for assigning students efficiently to schools without weakly violating any priorities. Then tie breaking does not matter.

The second difficulty is that each strict resolution of  $\succeq$  contains an Ergin-cycle. Therefore, by Theorem 1, for all strict resolutions  $\succeq'$  of  $\succeq$ ,  $f^{\succeq'}$  is not efficient. However, applying the modified version of Gale's top trading cycle algorithm (where 1

owns  $a$ , 2 owns  $b$ , and 3 owns  $c$ , and say the order in which students are allowed to choose is 1, 2, 3, and 4) by Abdulkadiroğlu and Sönmez (1999) yields for each profile  $R$  an efficient assignment. This rule is actually a “(fixed endowment) hierarchical exchange rule” (Pápai, 2000) with the following endowment inheritance table (an *endowment inheritance table* is a strict priority structure)<sup>7</sup>:

$\succeq'_a$	$\succeq'_b$	$\succeq'_c$	$\succeq'_d$
1	2	3	1
2	1	1	2
3	3	2	3
4	4	4	4

Note that  $\succeq'$  is a strict resolution of  $\succeq$ . For each profile, the initial endowments are given by the first entries in the inheritance table  $\succeq'$ . At Step 1 student 1 is endowed with  $\{a, d\}$ , 2 with  $\{b\}$ , and 3 with  $\{c\}$ . Then we apply Gale’s top trading cycle algorithm where each student points to his most preferred assignment. Students belonging to a cycle are removed with their assignments and the endowments they leave behind are reallocated according to  $\succeq'$ . For example, if student 1 prefers being unassigned to being assigned to any school, then 1 leaves behind  $a$  and  $d$ . If 2 was not assigned in the first step, then 2 inherits  $a$  and  $d$  from 1 and at Step 2 student 2’s endowment contains  $a$ ,  $b$ , and  $d$ . Then again Gale’s stop trading cycle algorithm is applied and so on.

Since hierarchical exchange rule are group strategy-proof (Pápai, 2000), the rule proposed by Abdulkadiroğlu and Sönmez (1999) is efficient, group strategy-proof, and weakly respects  $\succeq$ . Later we will see that any such rule must be a selection from the Gale-Shapley correspondence, i.e. any such rule selects for each profile a strict resolution of  $\succeq$  and applies the DA-algorithm to this profile and the selected strict resolution. In Example 1 the selected strict resolution must depend on the profile because  $f^{\succeq'}$  is not efficient for all strict resolutions  $\succeq'$  of  $\succeq$ . ◁

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<sup>7</sup>We refer the reader to Example 4 in Pápai (2000) for a more detailed description.

## 4 Strongly Respecting Priority Structures

Obviously there are other ways how to define a cycle than the one proposed by Ergin (2002). We propose the definition of a weak cycle.

**Definition 4 (Weak cycle)** *Let  $\succeq$  be a priority structure and  $q$  be a list of quotas. Then  $\succeq$  contains a weak cycle if there are distinct  $a, b \in A$  and  $i, j, k \in N$  such that the following conditions are satisfied:*

*(WC) Weak cycle condition:  $i \succeq_a j \succeq_a k$  and  $k \succeq_b i$ .*

*(WS) Weak scarcity condition: There exist (possibly empty) disjoint sets  $N_a, N_b \subseteq N \setminus \{i, j, k\}$  such that  $N_a \subseteq W_a(j)$ ,  $N_b \subseteq W_b(i)$ ,  $|N_a| = q_a - 1$ , and  $N_b = q_b - 1$ .*

Any Ergin-cycle is a weak cycle whereas the converse is not true. For strict priority structures the definitions of weak cycle and Ergin-cycle are equivalent. Our first result characterizes the priority structures which some efficient rule strongly respects.<sup>8</sup> It generalizes Theorem 1 for rules which strongly respect an arbitrary priority structure.

**Theorem 2** *Let  $\succeq$  be a priority structure and  $q$  be a list of quotas. The following are equivalent:*

*(i) There exists an efficient rule which strongly respects  $\succeq$ .*

*(ii)  $F^\succeq$  is single-valued and efficient.*

*(iii)  $F^\succeq$  is single-valued and group strategy-proof.*

*(iv)  $F^\succeq$  is single-valued and  $F^\succeq$  has a consistent extension.*

*(v)  $\succeq$  does not contain any weak cycle and for all  $a \in A$ , there exists  $i \in N$  such that both  $|W_a(i)| \leq q_a$  and  $\succeq_a |_{N \setminus W_a(i)}$  is strict.*

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<sup>8</sup>All proofs are in the Appendix.

Note that for strict priority structures (i) of Theorem 2 is superfluous because by (b) of Proposition 1 the DA-algorithm is the only candidate for an efficient rule which strongly respects a strict priority structure. This is no longer true for arbitrary priority structures  $\succeq$ . If we would choose a strict resolution  $\succeq'$  of  $\succeq$ , then by Theorem 1,  $f^{\succeq'}$  is efficient if  $\succeq'$  does not contain any Ergin-cycle. However, by Theorem 2,  $f^{\succeq'}$  only strongly respects  $\succeq$  if the Gale-Shapley correspondence is single-valued, i.e. it is irrelevant how we break ties in  $\succeq$ . If (ii) holds, then for any strict resolution  $\succeq'$  of  $\succeq$  the rule  $f^{\succeq'}$  is efficient and strongly respects the priority structure. Then also (iii) and (iv) hold and  $f^{\succeq'}$  is group strategy-proof and has a consistent extension.<sup>9</sup>

## 5 Weakly Respecting Priority Structures

Next we focus on rules which weakly respect a priority structure. The following result is a generalization of Proposition 1.<sup>10</sup>

**Proposition 2** *For any priority structure  $\succeq$  and any profile  $R$ ,*

- (a) *for any strict resolution  $\succeq'$  of  $\succeq$  the assignment  $f^{\succeq'}(R)$  weakly respects  $\succeq$ , and*
- (b) *if an assignment  $\mu$  weakly respects  $\succeq$  and no other assignment, which weakly respects  $\succeq$ , is Pareto superior to  $\mu$ , then there exists a strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}(R) = \mu$ .*

Given a priority structure  $\succeq$ , an Ergin-cycle is a property of the strict orderings of  $\succeq$  whereas a weak cycle is a property of the weak orderings of  $\succeq$ . The following two cycle conditions are in between these two definitions in the sense that they are properties of both the strict and weak orderings of  $\succeq$ .

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<sup>9</sup>In Theorem 2 it is not possible to add the statement “there exists a group strategy-proof rule which strongly respects  $\succeq$ ” because the rule which leaves all students unassigned at any profile is group strategy-proof and strongly respects  $\succeq$ .

<sup>10</sup>Part (a) of Proposition 2 is informally described in Roth and Sotomayor (1990,p.30).

**Definition 5** Let  $\succeq$  be a priority structure.

(I) Then  $\succeq$  contains a Type-I cycle if there are distinct  $a, b \in A$  and  $i, j, k \in N$  such that the following conditions are satisfied:

(I-C) I-cycle condition:  $i \sim_a j \succ_a k$  and both  $k \succ_b i$  and  $k \succ_b j$ .

(I-S) I-scarcity condition: There exist (possibly empty) disjoint sets  $N_a, N_b \subseteq N \setminus \{i, j, k\}$  such that  $N_a \subseteq U_a(k)$ ,  $N_b \subseteq U_b(i) \cap U_b(j)$ ,  $|N_a| = q_a - 1$ , and  $|N_b| = q_b - 1$ .

(II) Then  $\succeq$  contains a Type-II cycle if there are distinct  $a, b, c \in A$  and  $i, j, k, l \in N$  such that the following conditions are satisfied:

(II-C) II-cycle condition:  $i \sim_a j \succ_a k$ ,  $i \sim_a j \succ_a l$ , and both  $k \succ_b i$  and  $l \succ_c j$ .

(II-S) II-scarcity condition: There exist (possibly empty) disjoint sets  $N_a, N_b, N_c \subseteq N \setminus \{i, j, k, l\}$  such that  $N_a \subseteq U_a(k) \cap U_a(l)$ ,  $N_b \subseteq U_b(i)$ ,  $N_c \subseteq U_c(j)$ ,  $|N_a| = q_a - 1$ ,  $|N_b| = q_b - 1$ , and  $|N_c| = q_c - 1$ .

It is straightforward that the priority structure  $\succeq$  in Example 1 contains neither Type-I cycles nor Type-II cycles because any tie is located at the “bottom” of a priority ordering. Our next result characterizes the efficient rules which weakly respect a priority structure.

**Theorem 3** Let  $\succeq$  be a priority structure,  $q$  be a list of quotas, and  $f$  be a rule. The following are equivalent:

(i)  $f$  is an efficient rule which weakly respects  $\succeq$ .

(ii)  $f$  is an efficient selection from  $F^\succeq$ .

Furthermore, if (i) or (ii) hold, then  $\succeq$  contains no Ergin-cycle, no Type-I cycle and no Type-II cycle.

Theorem 3 says that for any priority structure  $\succeq$  and any profile  $R$ , if there exists an efficient assignment which weakly respects  $\succeq$  under  $R$ , then this assignment must be obtained from applying the DA-algorithm to a strict resolution of  $\succeq$  and  $R$ .

Theorem 3 does not say that if there exists an efficient rule which weakly respects  $\succeq$ , then we can find a strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}$  is efficient. Example 1 shows that indeed this implication does not need to hold and that the hierarchical exchange rule based on  $\succeq'$  is an efficient selection from the Gale-Shapley correspondence.

Let  $\succeq$  be a priority structure. When pursuing the problem of finding efficient rules which weakly respect  $\succeq$  there are two plausible approaches. The first one is the “Gale-Shapley approach”: we select a strict resolution  $\succeq'$  of  $\succeq$  and check whether  $f^{\succeq'}$  is efficient. Here, by (a) of Proposition 2,  $f^{\succeq'}$  always weakly respects  $\succeq$ .

The second one is the “endowment inheritance approach”: we select a strict resolution  $\succeq'$  of  $\succeq$  and check whether the hierarchical exchange rule using  $\succeq'$  as the endowment inheritance table weakly respects  $\succeq$ . Here the hierarchical exchange rule using  $\succeq'$  is always efficient.

We show that neither the “Gale-Shapley approach” nor the “endowment inheritance approach” successfully solve assignment problems with a weak priority structure. It is possible that there exists an efficient and group strategy-proof rule which weakly respects the priority structure although neither a DA-algorithm nor a hierarchical exchange rule with a strict resolution of the priority structure provide an efficient rule which weakly respects the priority structure.

**Theorem 4** *There are priority structures  $\succeq$  and lists of quotas  $q$  such that*

- (i) *there exists no strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}$  is efficient;*
- (ii) *there exists no strict resolution  $\succeq'$  of  $\succeq$  such that the hierarchical exchange rule with endowment inheritance table  $\succeq'$  weakly respects  $\succeq$ ; and*
- (iii) *there exists an efficient and group strategy-proof rule which weakly respects  $\succeq$ .*

We will show Theorem 4 via the following example. Let  $N = \{1, 2, 3\}$ ,  $A = \{a, b, c\}$ , and  $q_a = q_b = q_c = 1$ . Let  $\succeq$  be the priority structure such that

$$\begin{array}{ccc} \succeq_a & \succeq_b & \succeq_c \\ \hline 1, 2 & 2, 3 & 1, 3 \\ 3 & 1 & 2 \end{array}$$

In Appendix C we will propose an efficient and group strategy-proof rule which weakly respects the priority structure  $\succeq$ . By Theorem 3, this rule must choose for each profile a Gale-Shapley assignment which is obtained from a strict resolution of  $\succeq$  and this profile. By (i) in Theorem 4, tie breaking cannot be exogenous, i.e. it is not possible to fix a priori a strict resolution  $\succeq'$  of  $\succeq$  and apply to each profile the DA-algorithm with fixed preferences  $\succeq'$ . The rule we will define is a modified version of the DA-algorithm which specifies for any profile a resolution of indifferences. Here the main idea is that if two students propose to the same school and they have equal priority at that school, then the tie is broken by the priority ordering of the school to which the third student proposes. For example, if 1 and 2 propose to school  $a$ , then we have  $1 \sim_a 2$  and the DA-algorithm “would not know” which student to reject. Now if 3 proposes to  $b$ , then  $\succeq_b$  breaks the tie  $1 \sim_a 2$  in favor of 2 and 1 is rejected by school  $a$ . If 3 proposes to  $c$ , then  $\succeq_c$  breaks the tie  $1 \sim_a 2$  in favor of 1 and 2 is rejected by school  $a$ . We refer the reader to Appendix C for a complete definition.

There are many other weak priority structures for which the “modified” DA-algorithm is superior to the “Gale-Shapley approach” and the “endowment inheritance approach”. For strict priority structures the “modified” DA-algorithm and the DA-algorithm are identical. Furthermore by Theorem 3 there exists a specification of endogenous tie breaking such that the “modified” DA-algorithm and the hierarchical exchange rule of Example 1 are identical.<sup>11</sup>

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<sup>11</sup>Here tie breaking is sequential. If two students, say 3 and 4, apply to  $a$ , then the priority ordering of the school 1 proposes to breaks the tie. In case 1 applies to  $b$ , then  $\succeq_b$  does not resolve the tie  $3 \sim_a 4$  and the priority ordering of the school 2 applies to breaks the tie. We omit the details.

Unfortunately Theorem 2 does not remain true for rules which weakly respect a priority structure. For instance, if an efficient rule weakly respects a priority structure, then this rule is not necessarily strategy-proof. For example, take a priority structure where for each school  $a \in A$ ,  $\succeq_a$  is complete indifference over all students. Then any efficient rule weakly respects  $\succeq$  but such a rule does not need to be strategy-proof. However, if we choose an acyclic resolution of this priority structure, then by Theorem 2 and 3, the DA-algorithm applied to this resolution is an efficient and strategy-proof rule which weakly respects this priority structure.

**Proposition 3** *There are priority structures  $\succeq$  and lists of quotas such that there exists an efficient and group strategy-proof rule which weakly respects  $\succeq$  but no efficient rule, which weakly respects  $\succeq$ , has a consistent extension.*

Proposition 3 shows in a strong sense that the relations between the properties in Theorem 2 no longer hold for rules which weakly respect priority structures: even if there is one which is both efficient and group strategy-proof, we may not be able to find an efficient rule which has a consistent extension.

## 6 Conclusion

Related and parallel research are Erdil and Ergin (2005) and Abdulkadiroğlu, Pathak and Roth (2006). Both of these papers consider rules weakly respecting a weak priority structure.

Erdil and Ergin (2005) propose a simple procedure in order to construct for any priority structure  $\succeq$  and any profile  $R$ , an assignment  $\mu$ , which weakly respects  $\succeq$ , and no other assignment, which weakly respects  $\succeq$ , is Pareto superior to  $\mu$ . Let us call such an assignment *student-optimal*. More precisely, starting from a strict resolution  $\succeq'$  of  $\succeq$ , they check whether  $f^{\succeq'}(R)$  is student-optimal or not. If  $f^{\succeq'}(R)$  is not student-optimal, then a “stable improvement cycle” exists along which students

exchange the schools assigned to them under  $f^{\succeq'}(R)$ . Iterative elimination of “stable improvement cycles” (by letting students exchange schools along these cycles) eventually yields a student-optimal assignment. By Proposition 2, their procedure determines for any  $\succeq$  and any  $R$  a strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}(R)$  is student-optimal. Erdil and Ergin (2005) also show that it is not always possible to find a strategy-proof and student-optimal selection from the Gale-Shapley correspondence. In other words, the DA-algorithm with endogenous tie-breaking in the example used to prove Theorem 4 cannot be generalized to arbitrary priority structures without losing student-optimality or strategy-proofness.

Contrary to this paper and Erdil and Ergin (2005), Abdulkadiroğlu, Pathak and Roth (2006) focus on fixed tie-breaking. More precisely, they fix a priori a ranking of the students according to which any ties are broken in  $\succeq$ . They generalize Proposition 2 and show that any student-optimal assignment can be obtained via applying the DA-algorithm to a strict resolution of  $\succeq$  where ties are broken according to an a priori fixed ranking of the students. More importantly, they show that the DA-algorithm with any fixed tie-breaking is not dominated (at all profiles) by any other strategy-proof mechanism. In other words, given a strict resolution  $\succeq'$  of  $\succeq$ , there does not exist any strategy-proof mechanism  $f$  (not identical with  $f^{\succeq'}$ ) such that for all profiles  $R$  the assignment  $f(R)$  is Pareto superior to  $f^{\succeq'}(R)$ . This means that in the example used to prove Theorem 4, neither the DA-algorithm with endogenous tie-breaking dominates the DA-algorithm with fixed tie-breaking nor that the DA-algorithm with fixed tie-breaking dominates the DA-algorithm with endogenous tie-breaking. Note, however, the DA-algorithm with fixed tie-breaking is not efficient whereas the DA-algorithm with endogenous tie-breaking is efficient.

## APPENDIX.

### Proof of Theorem 2:

**Lemma 1** *Let  $\succeq$  be a priority structure and  $f$  be a rule. If  $f$  strongly respects  $\succeq$ , then  $f$  strongly respects any strict resolution of  $\succeq$ .*

**Proof.** Let  $\succeq'$  be a strict resolution of  $\succeq$ . Suppose that  $f$  does not strongly respect  $\succeq'$ . Then there exist  $R \in \mathcal{R}$ ,  $i \in N$ , and  $a \in A$  such that  $f(R)$  weakly violates the priority of  $i$  for  $a$  at  $f(R)$ , i.e. for some  $j \in N$  we have  $f_j(R) = a$ ,  $aP_i f_i(R)$ , and  $i \succeq'_a j$ . Because  $\succeq'_a$  is strict and  $i \neq j$ , we have  $i \succ'_a j$ . Because  $\succeq'$  is a strict resolution of  $\succeq$ , it follows that

$$i \succeq_a j. \tag{1}$$

Since  $f$  strongly respects  $\succeq$ ,  $f(R)$  does not weakly violate the priority of  $i$  for  $a$  at  $f(R)$ . Thus,  $j \succ_a i$ , which contradicts (1).  $\square$

We prove Theorem 2 by showing (i) $\Leftrightarrow$ (ii) and (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (v).

(i) $\Rightarrow$ (ii): Let  $f$  be an efficient rule which strongly respects  $\succeq$ . Let  $\succeq'$  be a strict resolution of  $\succeq$ . By Lemma 1,  $f$  strongly respects  $\succeq'$ . Thus,  $f$  is an efficient rule which strongly respects  $\succeq'$ . Because  $\succeq'$  is strict, Definitions 1 and 2 are equivalent. Thus,  $f$  weakly respects  $\succeq'$  and Proposition 1 implies  $f = f^{\succeq'}$ . Since  $\succeq'$  was an arbitrary strict resolution of  $\succeq$ ,  $F^{\succeq}$  is single-valued and efficient.

(ii) $\Rightarrow$ (i): Let  $\succeq'$  be a strict resolution of  $\succeq$  and  $f \equiv f^{\succeq'}$ . By (ii),  $f$  is well-defined and efficient. Suppose that  $f$  does not strongly respect  $\succeq$ . Then for some  $R \in \mathcal{R}$ , there are  $i \in N$  and  $a \in A$  such that  $f(R)$  weakly violates the priority of  $i$  for  $a$  at  $f(R)$ , i.e. for some  $j \in N$ ,  $f_j(R) = a$ ,  $aP_i f_i(R)$ , and  $i \succeq_a j$ . Let  $\succeq''$  be a strict resolution of  $\succeq$  such that  $i \succ''_a j$ . By (ii),  $f = f^{\succeq'} = f^{\succeq''}$ . Because  $f^{\succeq''}$  strongly respects  $\succeq''$ , this contradicts the facts that  $f(R)$  weakly violates the priority of  $i$  for  $a$  at  $f(R)$  and  $f(R) = f^{\succeq''}(R)$ .

(v) $\Rightarrow$ (iv): Let  $\succeq'$  be a strict resolution of  $\succeq$ . Because  $\succeq$  does not contain any weak cycle, it follows that  $\succeq'$  does not contain any weak cycle. Because  $\succeq'$  is strict

and the definitions of Ergin-cycle and weak cycle are equivalent for strict priority structures,  $\succeq'$  does not contain any Ergin-cycle. Hence, by Theorem 1,  $f^{\succeq'}$  has a consistent extension.

Let  $\succeq'$  and  $\succeq''$  be strict resolutions of  $\succeq$  and  $R \in \mathcal{R}$ . Let  $a \in A$ . By (v) there exists  $i \in N$  such that  $|W_a(i)| \leq q_a$  and  $\succeq_a \upharpoonright_{N \setminus W_a(i)}$  is strict. Thus,  $\succeq'_a \upharpoonright_{N \setminus W_a(i)} = \succeq''_a \upharpoonright_{N \setminus W_a(i)}$ . Because  $|W_a(i)| \leq q_a$ , each student belonging to  $W_a(i)$  is accepted by  $a$  if he proposes to school  $a$  when applying the DA-algorithm to  $(\succeq', R)$  or  $(\succeq'', R)$ . Therefore,  $f^{\succeq'}(R) = f^{\succeq''}(R)$  and  $F^\succeq$  is single-valued.

(iv)  $\Rightarrow$  (iii): Let  $\succeq'$  be a strict resolution of  $\succeq$ . By (iv),  $F^\succeq$  is single-valued and  $F^\succeq$  has a consistent extension. Thus,  $F^\succeq(R) = \{f^{\succeq'}(R)\}$  for all  $R \in \mathcal{R}$  and  $f^{\succeq'}$  has a consistent extension. Hence, by Theorem 1,  $f^{\succeq'}$  is group strategy-proof.

(iii)  $\Rightarrow$  (ii): Let  $\succeq'$  be a strict resolution of  $\succeq$ . By (iii),  $F^\succeq$  is single-valued and group strategy-proof. Thus,  $F^\succeq(R) = \{f^{\succeq'}(R)\}$  for all  $R \in \mathcal{R}$  and  $f^{\succeq'}$  is group strategy-proof. Hence, by Theorem 1,  $f^{\succeq'}$  is efficient.

(ii)  $\Rightarrow$  (v): Suppose that  $\succeq$  contains a weak cycle, i.e. there are  $a, b \in A$  and  $i, j, k \in N$  such that  $i \succeq_a j \succeq_a k \succeq_b i$ , and there exist (possibly empty) disjoint sets  $N_a, N_b \subseteq N \setminus \{i, j, k\}$  such that  $N_a \subseteq W_a(j)$ ,  $N_b \subseteq W_b(i)$ ,  $|N_a| = q_a - 1$ , and  $N_b = q_b - 1$ . Let  $\succeq'$  be a strict resolution of  $\succeq$  such that  $i \succ'_a j \succ'_a k \succ'_b i$ ,  $N_a \subseteq U_a(j)$ , and  $N_b \subseteq U_b(i)$ . Then  $\succeq'$  contains an Ergin-cycle. Hence, by Theorem 1,  $f^{\succeq'}$  is not efficient. Since  $F^\succeq(R) = \{f^{\succeq'}(R)\}$  for all  $R \in \mathcal{R}$ , this contradicts (ii).

Let  $a \in A$ . We show that there exists  $i \in N$  such that  $|W_a(i)| \leq q_a$ . Let  $R \in \mathcal{R}$  be such that for all  $i \in N$  and all  $b \in A$ ,  $aR_i bR_i i$ . If for all  $i \in N$ ,  $|W_a(i)| > q_a$ , then there exist two strict resolutions  $\succeq'$  and  $\succeq''$  of  $\succeq$  such that  $f^{\succeq'}(R) \neq f^{\succeq''}(R)$ , a contradiction to (ii). Similarly, it can be shown that  $\succeq_a \upharpoonright_{N \setminus W_a(i)}$  must be strict.  $\square$

## Proof of Proposition 2:

Let  $\succeq$  be a priority structure and  $R$  be a profile.

(a): Let  $\succeq'$  be a strict resolution of  $\succeq$ . Suppose the assignment  $f^{\succeq'}(R)$  does not weakly respect  $\succeq$ . Then there are  $i \in N$ , and  $a \in A$  such that  $f^{\succeq'}(R)$  strongly

violates the priority of  $i$  for  $a$  at  $f^{\succeq'}(R)$ , i.e. for some  $j \in N$ ,  $f_j^{\succeq'}(R) = a$ ,  $aP_i f_i^{\succeq'}(R)$ , and  $i \succ_a j$ . Because  $\succ'_a$  preserves  $\succ_a$ , we have  $i \succ'_a j$ . Thus,  $f^{\succeq'}(R)$  does not weakly respect  $\succeq'$ , a contradiction to (a) of Proposition 1.

(b): Let  $\mu$  be an assignment, which weakly respects  $\succeq$ , and no other assignment, which weakly respects  $\succeq$ , is Pareto superior to  $\mu$ . Let  $\succeq'$  be a strict resolution of  $\succeq$  such that for all  $a \in A$  and all  $i, j \in N$ , if  $\mu(j) = a$ ,  $\mu(i) \neq a$ , and  $j \succeq_a i$ , then  $j \succ'_a i$ . We show that  $\mu$  weakly respects  $\succeq'$  under  $R$ .

Suppose that there exist  $i \in N$  and  $a \in A$  such that  $\mu$  strongly violates the priority of  $i$  for  $a$  at  $\mu$ , i.e. for some  $j \in N$ ,  $\mu(j) = a$ ,  $aP_i \mu(i)$ , and  $i \succ'_a j$ . Thus,  $\mu(i) \neq a$  and by our choice of  $\succeq'_a$ , we must have  $i \succ_a j$ . Hence,  $\mu$  does not weakly respect  $\succeq$  under  $R$ , a contradiction.

By (a), we know that  $f^{\succeq'}(R)$  weakly respects  $\succeq$ . Since  $f^{\succeq'}(R)$  is not Pareto superior to  $\mu$  and both  $f^{\succeq'}(R)$  and  $\mu$  weakly respect  $\succeq'$ , (b) of Proposition 1 implies  $f^{\succeq'}(R) = \mu$ , the desired conclusion.  $\square$

### Proof of Theorem 3:

(i) $\Rightarrow$ (ii): Let  $f$  be an efficient rule which weakly respects  $\succeq$  and  $R \in \mathcal{R}$ . By (i),  $f(R)$  is an efficient assignment which weakly respects  $\succeq$  under  $R$ . Thus, by (b) of Proposition 2, there exists a strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}(R) = f(R)$ , the desired conclusion.

(ii) $\Rightarrow$ (i): Let  $R \in \mathcal{R}$ . By (ii), there exists a strict resolution  $\succeq'$  of  $\succeq$  such that  $f^{\succeq'}(R)$  is efficient. Define  $f(R) \equiv f^{\succeq'}(R)$ . By (a) of Proposition 2,  $f$  is an efficient rule which weakly respects  $\succeq$ , the desired conclusion.

**Lemma 2** *Let  $\succeq$  be a priority structure. If (i) or (ii) hold, then  $\succeq$  contains no Ergin-cycle, no Type-I cycle and no Type-II cycle.*

**Proof.** If  $\succeq$  contains an Ergin-cycle or a Type-I cycle, then the proof is similar to Ergin (2002) and left to the reader. Suppose that  $\succeq$  has a Type-II cycle, i.e. there

are distinct  $a, b, c \in A$  and  $i, j, k, l \in N$  such that (II-C)  $i \sim_a j \succ_a k$ ,  $i \sim_a j \succ_a l$ , and both  $k \succ_b i$  and  $l \succ_c j$ , and (II-S) there exist (possibly empty) disjoint sets  $N_a, N_b, N_c \subseteq N \setminus \{i, j, k, l\}$  such that  $N_a \subseteq U_a(k) \cap U_a(l)$ ,  $N_b \subseteq U_b(i)$ ,  $N_c \subseteq U_c(j)$ ,  $|N_a| = q_a - 1$ ,  $|N_b| = q_b - 1$ , and  $|N_c| = q_c - 1$ .

We show that there exists no efficient rule which weakly respects  $\succeq$ , i.e. neither (i) nor (ii) hold. Suppose that  $f$  is such a rule. Let  $R \in \mathcal{R}$  be such that

$R_i$	$R_j$	$R_k$	$R_l$
$b$	$c$	$a$	$a$
$c$	$b$	$b$	$c$
$a$	$a$		

all students in  $N_a$ ,  $N_b$  and  $N_c$  respectively rank  $a$ ,  $b$ , and  $c$  as the only acceptable school, and all other agents prefer being not assigned to being assigned to some school. Because  $|N_a| = q_a - 1$ ,  $N_a \subseteq U_a(k) \cap U_a(l)$ ,  $i \sim_a j \succ_a k$  and  $i \sim_a j \succ_a l$ , we cannot have  $f_i(R) = i$  or  $f_j(R) = j$ . If  $\{f_i(R), f_j(R)\} = \{b, c\}$ , then by efficiency,  $f_i(R) = b$  and  $f_j(R) = c$ . Since  $N_b \subseteq U_b(i)$  and  $N_c \subseteq U_c(j)$ , we have  $f_h(R) = b$  for all  $h \in N_b$  and  $f_h(R) = c$  for all  $h \in N_c$ . Since  $|N_b| = q_b - 1$  and  $|N_c| = q_c - 1$ ,  $f_k(R) \neq b, c$  and  $f_l(R) \neq b, c$ . If  $f_k(R) = k$ , then  $k$  envies  $i$  and  $k \succ_b i$ , a contradiction. If  $f_l(R) = l$ , then  $l$  envies  $j$  and  $l \succ_c j$ , a contradiction. If  $f_i(R) = a$  or  $f_j(R) = a$ , then a Pareto improvement is possible, a contradiction.  $\square$

### Proof of Theorem 4:

Recall that  $N = \{1, 2, 3\}$ ,  $A = \{a, b, c\}$ ,  $q_a = q_b = q_c = 1$ , and  $\succeq$  is such that

$\succeq_a$	$\succeq_b$	$\succeq_c$
1, 2	2, 3	1, 3
3	1	2

(i): Let  $\succeq'$  be a strict resolution of  $\succeq$ . If  $1 \succ'_a 2$ , then by  $3 \succ_b 1$  and  $q_a = q_b = 1$ ,  $1 \succ'_a 2 \succ'_a 3 \succ'_b 1$  is an Ergin-cycle. If  $2 \succ'_a 1$ , then by  $3 \succ_c 2$  and  $q_a = q_c = 1$ ,  $2 \succ'_a 1 \succ'_a 3 \succ'_c 2$  is an Ergin-cycle. Thus, by Theorem 1,  $f^{\succeq'}$  is not efficient.

(ii): Let  $\succeq'$  be a strict resolution of  $\succeq$ . We denote the (fixed endowment) hierarchical exchange rule with endowment inheritance table  $\succeq'$  by  $h^{\succeq'}$ . Because  $\succeq'_a$  breaks the ties in  $\succeq_a$ , we have  $1 \succ'_a 2 \succ'_a 3$  or  $2 \succ'_a 1 \succ'_a 3$ . Suppose  $1 \succ'_a 2 \succ'_a 3$ . Because  $\succeq'_b$  breaks the ties in  $\succeq_b$ , we have  $2 \succ'_b 3 \succ'_b 1$  or  $3 \succ'_b 2 \succ'_b 1$ . Suppose  $2 \succ'_b 3 \succ'_b 1$ . Let  $R \in \mathcal{R}$  be such that  $bP_11P_1aP_1c$ ,  $aP_22P_2bP_2c$ , and  $bP_3cP_33P_3a$ . At the first step of Gale's top trading cycle algorithm, 1 points to  $b$  and 2 points to  $a$ . Thus, 1 and 2 exchange their endowments. We have  $h_1^{\succeq'}(R) = b$ ,  $h_2^{\succeq'}(R) = a$ , and  $h_3^{\succeq'}(R) \neq b$ . Hence,  $h_1^{\succeq'}(R) = b$ ,  $bP_3h_3^{\succeq'}(R)$ , and  $3 \succ_b 1$ , i.e.  $h^{\succeq'}(R)$  strongly violates the priority of 3 for  $b$  and  $h^{\succeq'}$  does not weakly respect  $\succeq$ .

(iii): Let  $R \in \mathcal{R}$ . For all  $i \in N$ , we denote by  $t(R_i) \in A \cup \{i\}$  the top ranked assignment under  $R_i$  and by  $s(R_i) \in A \cup \{i\}$  the second ranked assignment under  $R_i$ . We propose the following modified DA-algorithm, denoted by  $h$ : Let  $R \in \mathcal{R}$  and  $N = \{i, j, k\}$ . Let  $\succeq_0$  be the priority ordering such that  $1 \succ_0 2 \succ_0 3$ .

At the first step, each student applies to his favorite school. Each school  $x \in A$  places the student on the waiting list who has the highest priority for  $x$  under  $\succeq_x$  and the others are rejected. If two students have the same priority under  $\succeq_x$ , say  $i \sim_x j$ , then using  $R_k$  the tie is broken as follows:

(i) if  $t(R_k) \in A \setminus \{x\}$ , then  $\succeq_{t(R_k)}$  breaks the tie  $i \sim_x j$ . Note that under  $\succeq_{t(R_k)}$  we cannot have  $i \sim_{t(R_k)} j$ , i.e. the tie  $i \sim_x j$  is either broken in favor of  $i$  or  $j$ .

(ii) if  $t(R_k) = x$  and  $s(R_k) \in A$ , then  $\succeq_{s(R_k)}$  breaks the tie  $i \sim_x j$ .

(iii) otherwise (this is the case when  $t(R_k) = k$  or  $[t(R_k) = x$  and  $s(R_k) = k]$ )  $\succeq_0$  breaks the tie  $i \sim_x j$ .

At the second step, those applicants who were rejected at Step 1 apply to their next best acceptable schools. Each school  $x \in A$  places the student on the waiting list who has the highest priority for  $x$  under  $\succeq_x$  among the new applicants and the student in the waiting list. If two students have

the same priority under  $\succeq_x$ , say  $i \sim_x j$ , then  $\succeq_{t(R_k)}$  breaks the tie  $i \sim_x j$ . Note that since  $i$  or  $j$  has been rejected in Step 1 and  $i$  and  $j$  apply to the same school Step 2, we must have that in Step 1  $i$  or  $j$  has applied to the same school as  $k$  and  $t(R_k)$  rejected  $i$  or  $j$ .

At the third step, those applicants who were rejected at the previous step apply to their next best acceptable schools. Each school  $x \in A$  places the student on the waiting list who has the highest priority for  $x$  under  $\succeq_x$  among the new applicants and the student in the waiting list. If two students have the same priority under  $\succeq_x$ , say  $i \sim_x j$ , then  $\succeq_{s(R_k)}$  breaks the tie  $i \sim_x j$ .

At the fourth step, those applicants who were rejected at the previous step apply to their next best acceptable schools.

The algorithm terminates when every student is either on a waiting list or has been rejected by every school that is acceptable to him. After this procedure ends, each school admits the student on its waiting list which yields the “modified” Gale-Shapley assignment for the problem  $(\succeq, R)$ . We denote this assignment by  $h(R)$ . We show that the third step of  $h$  is well-defined.

**Lemma 3** *The modified DA-algorithm  $h$  has the following property: if two students  $i$  and  $j$  apply to school  $x$  at Step 3, then (i) student  $k$  applied to school  $x$  at Step 1 and applies to another school at Step 2, (ii)  $i \sim_x j$ , and (iii) at Step 3 school  $x$  rejects the new applicant, i.e. the tie  $i \sim_x j$  is broken in favor of the student who was on the waiting list of school  $x$  after Step 2.<sup>12</sup>*

**Proof.** Let student  $i$  be rejected at Step 2 and apply to school  $c$  in Step 3. Suppose that  $i$  was rejected by school  $b$  at Step 2 and school  $c$  has student  $j$  in its waiting list after Step 2. Then we have  $bP_i c P_i a$  (if  $aP_i b P_i c$ , then  $i$  would have been rejected

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<sup>12</sup>Note that Lemma 3 is not true for  $f^{\succeq'}$  (where  $\succeq'$  is a strict resolution of  $\succeq$ ).

in Steps 1 and 2 by  $a$  and  $b$ , respectively, and  $c$  cannot have a student in its waiting list). Because school  $c$  has student  $j$  in its waiting list after Step 2, students  $k$  and  $i$  apply to school  $b$  at Step 2. By  $bP_i c P_i a$ , we must have that  $j$  and  $k$  apply to school  $c$  in Step 1 and  $c$  rejects  $k$ . Thus,  $t(R_j) = t(R_k) = c$ ,  $s(R_k) = t(R_i) = b$ , and  $s(R_i) = c$ . In particular (i) is true for student  $k$ .

*Case 1:*  $\{j, k\} = \{1, 2\}$ .

Since school  $c$  rejects  $k$  at Step 1 and  $1 \succ_c 2$ , we must have  $j = 1$  and  $k = 2$ . Then at Step 2, students 2 and 3 apply to  $b$ . Since  $2 \sim_b 3$  and  $t(R_1) = c$ ,  $\succeq_c$  breaks the tie  $2 \sim_b 3$ . Hence, by  $3 \succ_c 2$ , 2 is rejected by school  $b$ . This contradicts the fact that  $i = 3$  is rejected at Step 2.

*Case 2:*  $\{j, k\} = \{2, 3\}$ .

Since school  $c$  rejects  $k$  at Step 1 and  $3 \succ_c 2$ , we must have  $j = 3$  and  $k = 2$ . Then at Step 2, students 1 and 2 apply to  $b$ . Hence, by  $2 \succ_b 1$ , 1 is rejected by school  $b$ . Then by  $bP_i c P_i a$  and  $i = 1$ , at Step 3 student 1 applies to school  $c$  and student 3 is on the waiting list of school  $c$  after Step 2. Because  $t(R_2) = c$  and  $s(R_2) = b$ ,  $\succeq_b$  breaks the tie  $1 \sim_c 3$  in favor of 3 and student  $i = 1$  is rejected by school  $c$  at Step 3. Hence, (ii) and (iii) are true.

*Case 3:*  $\{j, k\} = \{1, 3\}$ .

Since at Step 1 students 1 and 3 apply to school  $c$  and  $t(R_2) = b$ ,  $\succeq_b$  breaks the tie  $1 \sim_c 3$  in favor of 3. Because school  $c$  rejects  $k$  at Step 1, we must have  $j = 3$  and  $k = 1$ . Then at Step 2, students 1 and 2 apply to  $b$ . Hence, by  $2 \succ_b 1$ , 1 is rejected by school  $b$ . This contradicts the fact that  $i = 2$  is rejected at Step 2.  $\square$

Theorem 3 and the following lemma show that  $h$  is an efficient rule which weakly respects  $\succeq$ .

**Lemma 4** *The modified DA-algorithm  $h$  is an efficient selection from  $F^{\succeq}$ .*

**Proof.** Let  $R \in \mathcal{R}$ . By definition of the modified DA-algorithm, there exists a strict resolution  $\succeq'$  of  $\succeq$  such that  $h(R) = f^{\succeq'}(R)$ .

Suppose that  $h(R)$  is not efficient. By Theorem 2.27 in Roth and Sotomayor (1990),  $f^{\succeq'}(R)$  is weakly efficient.<sup>13</sup> Thus, by  $h(R) = f^{\succeq'}(R)$ , there exist two students, say  $i$  and  $j$  (where  $N = \{i, j, k\}$ ), who would like to exchange their assignments under  $h(R)$ . Then under the modified DA-algorithm  $i$  must have applied to  $h_j(R)$  and was rejected by  $h_j(R)$ , and  $j$  must have applied to  $h_i(R)$  and was rejected by  $h_i(R)$ . It is impossible that  $i$  was rejected by  $h_j(R)$  because  $j$  applied to  $h_j(R)$  and  $j$  was rejected by  $h_i(R)$  because  $i$  applied to  $h_i(R)$ . Hence, either  $i$  was rejected by  $h_j(R)$  because  $k$  applied to  $h_j(R)$  or  $j$  was rejected by  $h_i(R)$  because  $k$  applied to  $h_i(R)$ . Therefore, when applying the modified DA-algorithm  $h$  to  $R$ , there are at least three rejections. This means that at Step 3 student  $k$  is rejected and  $k$  belonged to the waiting list of a school after Step 2, which contradicts Lemma 3.  $\square$

The following lemma completes the proof of Theorem 4.

**Lemma 5** *The modified DA-algorithm  $h$  is group strategy-proof.*

**Proof.** Let  $R \in \mathcal{R}$ . If there is a unique efficient assignment under  $R$ , then for all  $i \in N$ ,  $h_i(R) = t(R_i)$  and no group of students can gain by manipulation. Let  $M \subseteq N$  and  $R' \equiv (R'_M, R_{N \setminus M})$ . Suppose that all students belonging to  $M$  weakly prefer  $h(R')$  to  $h(R)$  with strict preference holding for some student in  $M$ . Because  $h(R)$  is efficient, we have  $M \neq N$ . Since there are at least two efficient assignments under  $R$ , there are at least two students who have the same school top-ranked.

Let  $i \in N$  be such that  $t(R_i) \in A$  and at least one other student has also  $t(R_i)$  top-ranked under  $R$ . By definition of  $h$  and Lemma 3, one of the students who applies to  $t(R_i)$  at the first step is assigned to  $t(R_i)$  under  $h(R)$  (if two students apply to school  $t(R_i)$  at Step 3, then school  $t(R_i)$  keeps the student who was on its waiting list after Step 2). Without loss of generality, let  $h_i(R) = t_i(R)$ . Obviously  $i$  cannot gain by misrepresentation. We next show that  $i \notin M$ . If  $i \in M$ , then  $h_i(R') = h_i(R)$ .

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<sup>13</sup>Weak efficiency means that there is no assignment  $\mu$  such that all students strictly prefer  $\mu$  to  $f^{\succeq'}(R)$ .

Thus, by Lemma 3, either  $t(R'_i) = t(R_i)$  or  $[t(R'_i) \in A \setminus \{t(R_i)\}]$ ,  $i$  is rejected by  $t(R'_i)$  at the first step, and  $s(R'_i) = t(R_i)$ . By  $h_i(R') = h_i(R)$ , in both cases under  $R_i$  and  $R'_i$  ties are broken in the same way when applying the modified DA-algorithm to  $R$  and  $R'$ . Hence,  $i \notin M$ .

If  $h_i(R) = h_i(R')$  and  $R'_i = R_i$ , then by definition of  $h$  and Lemma 3, under  $R'$  we cannot have that  $M$  gains by joint manipulation. Thus,  $h_i(R') \neq h_i(R)$ . Let  $j \in N$  be such that  $h_j(R') = t(R_i)$ . Because  $h_i(R) = t(R_i)$  and at least one other student has  $t(R_i)$  top-ranked under  $R$ ,  $i$  cannot be the student at the bottom of the priority ordering  $\succeq_{t(R_i)}$ . Therefore,  $i \succeq_{t(R_i)} j$ . Because  $h_j(R') = t(R_i)$  and  $R'_i = R_i$ , we also have  $j \succeq_{t(R_i)} i$ . Thus,  $i \sim_{t(R_i)} j$ . Furthermore, by Lemma 3,  $R'_j$  must be such that either  $j$  applies to  $t(R_i)$  at the first step or  $j$  and  $k$  apply to the same school at the first step and  $j$  is rejected and applies to  $t(R_i)$  at the second step.

First suppose  $t(R'_j) = t(R_i)$ . Then the tie  $i \sim_{t(R_i)} j$  is broken according to (i), (ii), or (iii) in the first step of the modified DA-algorithm. If under  $R'$  the tie  $i \sim_{t(R_i)} j$  is broken according to  $\succeq_0$ , then  $h_k(R') = k$ . Since  $h_i(R) = t(R_i)$ , under  $R$  the tie  $i \sim_{t(R_i)} j$  cannot be broken according to  $\succeq_0$  and  $R'_k \neq R_k$ . But then  $h_k(R) \neq k$  and  $h_k(R)P_k k$ , which is a contradiction to  $R'_k \neq R_k$  and thus,  $k \in M$ . Similarly, if under  $R'$  the tie  $i \sim_{t(R_i)} j$  is broken according to  $\succeq_x$  where  $x \in A \setminus \{t(R_i)\}$ , then  $h_k(R') = x$ . Since  $h_i(R) = t(R_i)$ , under  $R$  the tie  $i \sim_{t(R_i)} j$  cannot be broken according to  $\succeq_x$  and  $R'_k \neq R_k$ . But then  $h_k(R) \neq x$  and  $h_k(R)P_k x$ , which is a contradiction to  $R'_k \neq R_k$  and thus,  $k \in M$ .

Second suppose that  $t(R'_j) = t(R'_k) \in A \setminus \{t(R_i)\}$ ,  $j$  is rejected by  $t(R'_k)$ , and  $s(R'_j) = t(R_i)$ . If  $k \succ_{t(R'_k)} j$ , then at the second step  $\succeq_{t(R'_k)}$  breaks the tie  $i \sim_{t(R_i)} j$  in favor of  $i$ , a contradiction. Thus,  $k \sim_{t(R'_k)} j$ . Since  $j$  is rejected by  $t(R'_k)$  under  $R'$ , we have  $h_k(R') = t(R'_k)$ . Since  $h_i(R) = t(R_i)$ , under  $R$  the tie  $i \sim_{t(R_i)} j$  cannot be broken according to  $\succeq_{t(R'_k)}$  and  $R'_k \neq R_k$ . But then  $h_k(R) \neq h_k(R')$  and  $h_k(R)P_k h_k(R')$ , which is a contradiction to  $R'_k \neq R_k$  and thus,  $k \in M$ .  $\square$

### Proof of Proposition 3:

Let  $N = \{1, 2, 3, 4\}$ ,  $A = \{a, b, c\}$ ,  $q_a = q_b = q_c = 1$ , and  $\succeq$  is such that

$\succeq_a$	$\succeq_b$	$\succeq_c$
4	4	4
1, 2	2, 3	1, 3
3	1	2

It is straightforward to adjust the proof of Theorem 4 and its modified DA-algorithm to  $\succeq$ . Hence, the modified DA-algorithm is an efficient and group strategy-proof rule which weakly respects  $\succeq$ . Let  $f$  be an efficient rule which weakly respects  $\succeq$ . We show  $f$  does not have a consistent extension. Let

$R_1$	$R_2$	$R_3$	$R_4$	$R'_4$
$b$	$c$	$a$	$c$	$a$
$c$	$b$	$b$	4	4
$a$	$a$	$c$	$b$	$b$
1	2	3	$a$	$c$

where  $R = (R_1, R_2, R_3, R_4)$  and  $R' = (R_{\{1,2,3\}}, R'_4)$ . Because  $f$  weakly respects  $\succeq$  and 4 has highest priority at each school, we have both  $f_4(R) = c$  and  $f_4(R') = a$ .

If  $f_3(R) = b$ , then  $f(R)$  is not efficient, a contradiction. Thus, by  $f_4(R) = c$  and both 1 and 2 have higher priority at school  $a$  than 3, we have  $f_1(R) \neq 1$  and  $f_2(R) \neq 2$ . Since  $f$  weakly respects  $\succeq$ , it follows  $f_1(R) = a$ ,  $f_2(R) = b$ , and  $f_3(R) = 3$ .

Similarly, if  $f_2(R') = b$ , then  $f(R')$  is not efficient, a contradiction. Thus, by  $f_4(R') = a$  and both 1 and 3 have higher priority at school  $c$  than 2, we have  $f_1(R') \neq 1$  and  $f_3(R') \neq 3$ . Since  $f$  weakly respects  $\succeq$ , it follows  $f_1(R') = c$ ,  $f_2(R') = 0$ , and  $f_3(R') = b$ .

Now we have  $\{f_1(R), f_4(R)\} = \{a, c\} = \{f_1(R'), f_4(R')\}$  and  $R_{\{2,3\}} = R'_{\{2,3\}}$ . However,  $f_2(R) = b \neq 2 = f_2(R')$  and  $f$  does not have a consistent extension.  $\square$

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