Nearly Serial Sharing Methods

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Abstract

A group of agents participate in a cooperative enterprise producing a single good. Each participant contributes a particular type of input; output is nondecreasing in the input profile. How should it be shared?

We analyze the implications of the axiom of Group Monotonicity: if a group of agents simultaneously decrease their input, not all of them should receive a bigger share of output. We show that in combination with other more familiar axioms, this condition pins down a very small class of methods, which we dub nearly serial.

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1 Introduction

A group of agents participate in a cooperative enterprise producing a single good (which we may think of as money). Each participant contributes a possibly different type of input; total output is a nondecreasing function of the input profile. How should this total output (or gross profit) be shared? We search for a simple method that would compute output shares as a function of the input profile and the production function.

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The case where inputs are either zero or one –each agent merely chooses whether to participate in the enterprise or not– corresponds to the model of cooperative games. In this much studied framework, the Shapley value (Shapley, 1953) stands out as the central sharing method. It is characterized by the three simple properties of Additivity, Dummy, and Anonymity: output shares depend additively on the production function, totally unproductive agents receive nothing, and equally productive agents get the same share.

We are interested here in the more complex problem where each participant may choose any level of input. Differences in output shares should now reflect not only differences in productivity, as the Shapley value does in the simpler case, but also differences in input levels. For instance, if two agents are equally productive, the one who contributes more should get a bigger share of output.

It is not surprising that in this richer model, Shapley's three axioms, properly reformulated, no longer characterize a unique method: there are different ways of combining the productivity and the sheer quantity of an input so as to compute the output share it deserves. Three prominent methods emerge from the literature. The Shapley-Shubik method (Shubik, 1962) applies the Shapley value to the so-called stand-alone game in which the worth of a coalition is the output generated by the input profile of its members. The Aumann-Shapley method, as adapted from Aumann and Shapley (1974) by Billera and Heath (1982) and Mirman and Tauman (1982), gives each agent the integral of his marginal product along the ray from zero to the input profile. The serial method (Friedman and Moulin, 1999) integrates the marginal product of each agent along the constrained-diagonal path to the input profile.[1] We refer to Friedman and Moulin (1999) for a comparison of these three methods. Moulin (2002) offers a general survey, including a discussion of the (suitable reformulation of the) above methods in the case where inputs come in indivisible units; see also Moulin (1995) and Moulin and Sprumont (2007).

In order to evaluate the relative merits of these and other methods, it is useful to formulate further axioms. An early example in the literature is Scale Invariance, stating that output shares should not depend on the units of

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[1] The constrained-diagonal (or constrained-egalitarian) path to an input profile x is the projection on $[0, x]$ of the diagonal path $z_i = z_j$ for all i, j. The method is called serial because it delivers the well-known serial formula of Moulin and Shenker (1992, 1994) when all inputs are perfect substitutes, that is, when total output depends only on the sum of the quantities supplied.

measurement of inputs. This axiom is central in the classic characterizations of the Aumann-Shapley method (Billera and Heath, 1982, and Mirman and Tauman, 1982).

The very nature of the variable-input model suggests conditions linking output shares in problems with different input profiles. Perhaps the simplest such condition is Monotonicity: a participant's share of output should be nondecreasing in his input. Even though preferences are not an explicit component of our model, Monotonicity may be interpreted as an incentive-compatibility condition. Any agent whose preferences are increasing in output and decreasing in his own input would have an incentive to manipulate a method that fails to be monotonic. Moulin (1995), who introduces the axiom, observes that it is satisfied by the Shapley-Shubik and the serial methods but is violated by the Aumann-Shapley method.

Moulin and Sprumont (2005) strengthen Monotonicity by requiring that a strict increase in all components of the input profile of a group of agents should not lead to a strict decrease in the output share of each of them. This is Group Monotonicity. A violation of this axiom leads to the possibility of strategic coordinated input deflation: all agents in some group could get a strictly higher share of output by agreeing to all contribute less input. In a cooperative environment where communication between agents is easy, Group Monotonicity is a compelling incentive-compatibility condition. Moulin and Sprumont (2005) show that the Shapley-Shubik method violates this condition, while the serial method satisfies it.

The purpose of this paper is to identify which methods meeting Shapley's three basic axioms satisfy Group Monotonicity. We work with the discrete version of the output-sharing model. Each agent i's input is an integer x_i; given the input profile x and the production function F, our method must allocate the total output $F(x)$.

The best known methods meeting Additivity and Dummy are the *path methods*. Fix an input profile x and consider a sequence $\{z^t\}$ from zero to x where each z^t is obtained from its predecessor by increasing the input of exactly one agent by one unit. For any production function F, compute any agent's output share by summing his marginal products along this sequence: agent i thus receives $\sum(F(z^t) - F(z^{t-1}))$ where the sum is taken over those t for which z^t obtains from z^{t-1} by increasing i's input by one unit. Under the *fixed-path methods*, the paths used for different input profiles x, x' are related: they obtain by projection of a single unbounded path in input space. The so-called *fixed-flow methods* introduced in Moulin and Sprumont (2005) are essentially the convex combinations of fixed-path methods.

Our first result is an axiomatization of the latter methods. We use a new powerful variant of the Dummy axiom, Irrelevance of Dummy Changes, stating that if the productivity of an agent's input is zero beyond a certain level, any increase beyond that level should leave the output shares unchanged. Theorem 1 asserts that Additivity, Irrelevance of Dummy Changes, and the mild requirement of Zero Output for Zero Input characterize the fixed-flow methods.

Building on this result, we explore the implications of Group Monotonicity. As already mentioned, an important example of a group-monotonic method is the serial method. In our discrete model, all the unbounded paths staying as close as possible to the line $z_i = z_j$ for all i, j qualify as "diagonal"; the serial method (Moulin, 1995) obtains by averaging the corresponding "diagonal" fixed-path methods. Theorem 2 states that the fixed-flow methods satisfying Anonymity and Group Monotonicity are, in a sense that will be made precise, *nearly serial*: they average fixed-path methods that are all *nearly diagonal*. Thus our axioms essentially characterize the serial method. We conjecture that an exact characterization holds in the continuous case.

A final word is in order about related contributions. Axiomatizations of various methods of the serial family exist in the literature. Moulin and Shenker (1994) characterize the serial formula for the case of perfectly substitutable inputs using an axiom placing upper bounds on output shares. With the aid of a similar axiom, Moulin (1995) and Friedman and Moulin (1999) characterize the serial method in the discrete and continuous contexts respectively. As noted in Moulin and Sprumont (2006), the Upper Bound axiom is intuitively reminiscent of the very serial formula. By contrast, none of the axioms used in Theorem 2 is directly related to a serial-type formula. Finally, Moulin and Sprumont (2006) offer an axiomatization of the serial method based on the property of Distributivity, which states that the sharing method should commute with the composition of production functions. This mathematical property akin to Additivity has no clear normative or strategic interpretation. By contrast, our main axiom, Group Monotonicity, is meaningful on both counts.

2 The model

Each agent i in a finite set $N = \{1, ..., n\}$ contributes an integer quantity $x_i \in \mathbb{N} = \{0, 1, 2, ...\}$ of a personalized input. The output generated by the *input profile* $x \in \mathbb{N}^N$ must be split among the members of N. A *production function* is a mapping $F : \mathbb{N}^N \to \mathbb{R}_+$ that is nondecreasing and satisfies $F(0) = 0$; the set of such mappings is denoted $\mathcal{F}(N)$. A (*output-sharing*) *method* (*for* N) is a mapping φ which assigns to each *problem* $(F, x) \in \mathcal{F}(N) \times \mathbb{N}^N$ a vector of nonnegative *output shares* $\varphi(F, x) \in \mathbb{R}_+^N$ satisfying the budget balance condition $\sum_{i \in N} \varphi_i(F, x) = F(x)$.

An alternative interpretation of this model is to regard x as a demand profile and F as a cost function: φ is then a cost-sharing method. We prefer the output-sharing interpretation because we find one of our axioms in Section 3, Irrelevance of Dummy Changes, more relevant in that context.

We use the following notation. The set of extended natural numbers is $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. Vector inequalities are written $\leq, <, \ll$. For all $x \in \mathbb{N}^N$ and $x' \in \overline{\mathbb{N}}^N$, $[x, x'] = \{z \in \mathbb{N}^N \mid x \leq z \leq x'\}$ and $]x, x'] = [x, x'] \backslash \{x\}$. For all $S \subseteq N$, we denote by $x_S \in \mathbb{N}^S$ the restriction of x to S and write $x(S) = \sum_{i \in S} x_i$. We sometimes write i for $\{i\}$, ij for $\{i, j\}$, and $-S$ for $N \backslash S$. We define $e^S \in \mathbb{N}^N$ by $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ otherwise. If $i \in N$ and $F \in \mathcal{F}(N)$, we define $\partial_i F : \mathbb{N}^N \to \mathbb{R}_+$ by $\partial_i F(z) = F(z) - F(z - e^i)$ whenever $z_i > 0$ and, by convention, $\partial_i F(z) = 0$ whenever $z_i = 0$.

3 The fixed-flow methods

This section proposes a simple axiomatization of the fixed-flow methods introduced in Moulin and Sprumont (2005). The first two axioms are well known, the third is new.

Additivity. For all $F, F' \in \mathcal{F}(N)$ and $x \in \mathbb{N}^N$, $\varphi(F + F', x) = \varphi(F, x) + \varphi(F', x)$.

This powerful mathematical property is very convenient in practice. Consider an enterprise producing several goods which it sells on a market. The final good to be shared among the participants is money and the total amount is the sum of the sales of the different goods. One may apply the sharing method directly to the aggregated sales or use it to divide the sales of each good and then add up the resulting shares. If the method is additive, the two procedures are equivalent. This is useful from an incentive viewpoint because the proper level of application of the method is not a matter of dispute.

Our second axiom says that an agent who contributes nothing should receive nothing.

Zero Output for Zero Input. For all $F \in \mathcal{F}(N)$, $x \in \mathbb{N}^N$, and $i \in N$, $\{x_i = 0\} \Rightarrow \{\varphi_i(F, x) = 0\}$.

Our third axiom is inspired by the traditional Dummy axiom according to which a "dummy", that is, a completely unproductive agent, should receive zero: for all $F \in \mathcal{F}(N)$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i F(z) = 0$ for all $z \in \mathbb{N}^N\} \Rightarrow \{\varphi_i(F, x) = 0\}$. Our condition says that "dummy changes" in inputs should have no effect on output shares.

Irrelevance of Dummy Changes. For all $F \in \mathcal{F}(N)$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i F(z) = 0$ for all z such that $z_i > x_i\} \Rightarrow \{\varphi(F, (z_i, x_{-i})) = \varphi(F, x)$ for all $z_i > x_i\}$.

Taken together, Zero Output for Zero Input and Irrelevance of Dummy Changes imply Dummy. In fact, they deliver a stronger property known as Strong Dummy: for all $F \in \mathcal{F}(N)$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i F(z) = 0$ for all $z \in \mathbb{N}^N\} \Rightarrow \{\varphi_i(F, x) = 0$ and $\varphi_j(F, x) = \varphi_j(F, (0_i, x_{-i}))$ for all $j \in N \backslash i\}$. This says that a dummy agent gets zero and that changes in his input do not affect others' output shares.

In order to describe the methods meeting Additivity, Zero Output for Zero Input, and Irrelevance of Dummy Changes, we use Moulin and Vohra's (2003) characterization of the methods satisfying Additivity and Dummy.

Definition 1. A (*unit*) *flow* to an input profile $x \in \mathbb{N}^N$ is a mapping $f(., x) : [0, x] \to \mathbb{R}_+^N$ such that $f_i(z, x) = 0$ if $z_i = 0$, $\sum_{i \in N(0,x)} f_i(e^i, x) = 1$, and

$$\sum_{i \in N} f_i(z, x) = \sum_{i \in N(z,x)} f_i(z + e^i, x) \text{ for all } z \in]0, x], \tag{1}$$

where $N(z, x) = \{i \in N | z_i < x_i\}$. Conditions (1) are the so-called flow conservation constraints. A *flow system* is a list $f = \{f(., x) | x \in \mathbb{N}^N\}$, where each $f(., x)$ is a flow to x.

Lemma 1 (Moulin and Vohra, 2003). *Let N be an arbitrary finite set of agents. An output-sharing method φ for N satisfies Additivity and Dummy* *if and only if there is a flow system f such that*

$$\varphi_i(F,x) = \sum_{z\in[0,x]} f_i(z,x)\partial_i F(z) \textit{for all } F \in \mathcal{F}(N),\ x \in \mathbb{N}^N, \textit{and } i \in N. \quad (2)$$

This system f is unique.

We say that the flow system f represents φ. Notice that all methods characterized in Lemma 1 guarantee Zero Output for Zero Input. Observe also that flows to different input profiles need not be related.

We will be concerned with flow systems where the flow to an input profile is simply the projection of the flow to any higher input profile. We define the projection operator in full generality as this will be useful in the proof of Theorem 2. Let $\underline{z} \in \mathbb{N}^N$, $\overline{z} \in \overline{\mathbb{N}}^N$, $\underline{z} \leq \overline{z}$. For all $T \subseteq N$, any mapping $h : [\underline{z}, \overline{z}] \to \mathbb{R}^T$ and all $x \in [\underline{z}, \overline{z}]$, the projection of h on $[\underline{z}, x]$ is the mapping $p_{[\underline{z},x]}h : [\underline{z}, x] \to \mathbb{R}^T$ defined as follows. For all $i \in T$ and $z \in [\underline{z}, x]$, write $K_i(z) = \{j \in N\backslash i \mid z_j = x_j\}$ and let

$$\begin{aligned}(p_{[\underline{z},x]}h)_i(z) &= h_i(z) \text{ if } K_i(z) = \emptyset,\\ &= \sum_{w_{K_i(z)}\in[x_{K_i(z)},\overline{z}_{K_i(z)}]} h_i(w_{K_i(z)}, z_{-K_i(z)}) \text{ otherwise.}\end{aligned}$$

Note that if $f(.,x)$ is a flow to x and $x' \leq x$, then $p_{[0,x']}f(.,x)$ is a flow to x'.

Definition 2. A *fixed flow system* is a flow system f such that

$$\{x' \leq x\} \Rightarrow \{f(.,x') = p_{[0,x']}f(.,x)\} \text{ for all } x, x' \in \mathbb{N}^N. \quad (3)$$

A method φ is a *fixed-flow method* if it is represented by a fixed flow system.

The fixed-flow methods were defined in Moulin and Sprumont (2005) under the restriction that input profiles are bounded above by some $\overline{x} \in \mathbb{N}^N$. In that case, the single flow $f(.,\overline{x})$ and the projection property (3) fully determine the entire flow system f. In our unbounded version, the flow $f(.,x)$ to any input profile x completely determines the flows to all input profiles $x' \leq x$. In particular, the system f is completely determined by the subsystem $\{f(.,ke^N)|k \in \mathbb{N}\}$.

Theorem 1. *Let N be an arbitrary finite set of agents. An output-sharing method φ for N satisfies Additivity, Zero Output for Zero Input, and Irrelevance of Dummy Changes if and only if φ is a fixed-flow method.*

Proof. The "if" statement is easily checked. To prove the converse statement, fix a method φ meeting the three axioms. Because Zero Output for Zero Input and Irrelevance of Dummy Changes imply Dummy, Lemma 1 applies: let f be the unique flow system representing φ through formula (2). We show that f is a fixed flow system.

In order to establish property (3), it is enough to prove that

$$f(.,x) = p_{[0,x]}f(.,x+e^i) \text{ for all } x \in \mathbb{N}^N \text{ and } i \in N. \tag{4}$$

This is because $p_{[0,x'']}p_{[0,x']}f(.,x) = p_{[0,x'']}f(.,x)$ whenever $x'' \leq x' \leq x$, as is clear from the definition of the projection operator.

The proof of (4) makes use of the following particular type of production function. For all $i \in N$ and $z \in \mathbb{N}^N$ such that $z_i > 0$, we define $z_0^j := z - z_i e^i + e^j$ if $j \in N \backslash i$, $z_0^i := z$, and let

$$F_z^i(w) = 1 \text{ if } w \geq z_0^j \text{ for some } j \in N, \text{ and 0 otherwise.} \tag{5}$$

By construction,

$$\partial_i F_z^i(w) = 1 \text{ if } w = z, \text{ and 0 otherwise,} \tag{6}$$

and

$$\partial_j F_z^i(w) = 0 \text{ whenever } w_j \geq z_j + 2. \tag{7}$$

We are now ready to prove (4). Fix $x \in \mathbb{N}^N$ and $i \in N$. The case $x = 0$ being trivial, assume $x \neq 0$. We make three claims.

Claim 1. *$f_i(z,x) = f_i(z,x+e^i)$ for all $z \in [0,x]$.*

To prove this claim, fix $z \in [0,x]$. If $z_i = 0$, $f_i(z,x) = f_i(z,x+e^i) = 0$ by definition: so we may assume $z_i > 0$. Consider the production function F_z^i. By definition, $\partial_i F_z^i(w) = 0$ whenever $w_i > x_i$. Thus, by Irrelevance of Dummy Changes, $\varphi(F_z^i, x+e^i) = \varphi(F_z^i, x)$. Focusing on agent i's output share and using (2) and (6), $f_i(z,x) = \varphi_i(F_z^i, x) = \varphi_i(F_z^i, x+e^i) = f_i(z, x+e^i)$.

Claim 2. *$f_j(z,x) = f_j(z,x+e^i)$ for all $j \in N \backslash i$ and all $z \in [0,x]$ such that $z_i < x_i$.*

Fix $j \in N \backslash i$ and $z \in [0,x]$ such that $z_i < x_i$. If $z_j = 0$, $f_j(z,x) = f_j(z,x+e^i) = 0$, so assume $z_j > 0$. Consider the production function F_z^j. By definition, $\partial_i F_z^j(w) = 0$ whenever $w_i \geq z_i + 2$. Since $z_i < x_i$, $\partial_i F_z^j(w) = 0$ whenever $w_i > x_i$. By Irrelevance of Dummy Changes, $\varphi(F_z^j, x+e^i) = \varphi(F_z^j, x)$. Focusing on agent j and using the representation formula (2), $f_j(z,x) = \varphi_j(F_z^j, x) = \varphi_j(F_z^j, x+e^i) = f_j(z, x+e^i)$.

Claim 3. *$f_j(z,x) = f_j(z,x+e^i) + f_j(z+e^i, x+e^i)$ for all $j \in N\backslash i$ and all $z \in [0,x]$ such that $z_i = x_i$.*

Fix $j \in N\backslash i$ and $z \in [0,x]$ such that $z_i = x_i$. Let F be the production function such that

$$\begin{aligned} \partial_i F(w) &= 0 \text{ for all } w \in \mathbb{N}^N, \text{ and} \\ \partial_j F(w) &= 1 \text{ if } w_{-i} = z_{-i}, \text{ and 0 otherwise.} \end{aligned}$$

This production function is easily constructed: define it first on those input profiles such that $z_i = 0$ by modifying the procedure in (5), then extend it to all profiles by setting $F(z) = F(0_i, z_{-i})$. By Irrelevance of Dummy Changes, $\varphi_j(F, x+e^i) = \varphi_j(F,x)$. Applying the representation formula (2),

$$\begin{aligned} \varphi_j(F,x) &= \sum_{w_i=0}^{x_i} f_j((w_i, z_{-i}), x), \\ \varphi_j(F, x+e^i) &= \sum_{w_i=0}^{x_i+1} f_j((w_i, z_{-i}), x+e^i). \end{aligned}$$

Taking Claim 2 into account, this means that $f_j((x_i, z_{-i}), x) = f_j((x_i, z_{-i}), x + e^i) + f_j((x_i+1, z_{-i}), x+e^i)$. Recalling that $z_i = x_i$, this means $f_j(z,x) = f_j(z, x+e^i) + f_j(z+e^i, x+e^i)$.

Claims 1, 2 and 3 together imply (4). ■

The axioms in Theorem 1 are independent. This follows directly from the independence of the axioms used in Theorem 2, which is established in the Section 5.

4 The nearly serial methods

Building on the previous section, we now show how two further axioms, Anonymity and Group Monotonicity, circumscribe the very small subclass of fixed-flow methods that we call *nearly serial.*

We begin by defining Anonymity. Denote by $\Pi(N)$ the set of bijections from N into itself and let $\pi \in \Pi(N)$. If $z \in \mathbb{R}_+^N$, define $\pi z \in \mathbb{R}_+^N$ by $(\pi z)_{\pi(i)} = z_i$ for all $i \in N$. If $F \in \mathcal{F}(N)$, define $\pi F \in \mathcal{F}(N)$ by $\pi F(\pi z) = F(z)$ for all $z \in \mathbb{N}^N$.

Anonymity. For all $\pi \in \Pi(N)$, $F \in \mathcal{F}(N)$, and $x \in \mathbb{N}^N$, $\varphi(\pi F, \pi x) = \pi\varphi(F, x)$.

This axiom expresses the familiar idea that the names of the agents should be ignored when computing the output shares: it is generally accepted as a basic notion of fairness. Our formulation implies weaker symmetry properties used in the literature: for instance, agents with equal input receive the same output share when the production function is symmetric.

Lemma 2. *Let N be an arbitrary finite set of agents. An output-sharing method φ for N satisfying Additivity and Dummy meets Anonymity if and only if its flow system f satisfies $f(\pi z, \pi x) = \pi f(z, x)$ for all $x \in \mathbb{N}^N$, $z \in [0, x]$, and $\pi \in \Pi(N)$.*

Proof. Sufficiency is clear from the representation formula (2). Conversely, suppose φ meets Additivity, Dummy and Anonymity; let f be its flow system. Fix $x \in N^N$, $z \in [0, x]$, and $\pi \in \Pi(N)$. Let $i \in N$ and consider the production function F_z^i defined in (5). By (2) and Anonymity, $f_i(z, x) = \varphi_i(F, x) = \varphi_{\pi(i)}(\pi F, \pi x) = f_{\pi(i)}(\pi z, \pi x)$.■

Focusing now on fixed-flow methods, let $k \in \mathbb{N}$ and consider the input profile ke^N where all agents contribute k units. We say that a flow $f(., ke^N)$ to ke^N is *anonymous* if

$$f(\pi z, ke^N) = \pi f(z, ke^N) \text{ for all } z \in [0, ke^N] \text{ and } \pi \in \Pi(N). \tag{8}$$

Taking into account the remark following Definition 2, it follows from Lemma 2 that a fixed-flow method φ meets Anonymity if and only if each flow $f(., ke^N)$, $k \in \mathbb{N}$, is anonymous. We then say that the fixed flow system f is anonymous.

We introduce now the serial and nearly serial methods. In order to do so, we need some further definitions.

Definition 3. Let $D^< = \{z \in \mathbb{N}^N \mid |z_i - \frac{z(N)}{n}| < 1 \text{ for all } i \in N\}$ and $D^\leq = \{z \in \mathbb{N}^N \mid |z_i - \frac{z(N)}{n}| \leq 1 \text{ for all } i \in N\}$; we call these sets the diagonal and near-diagonal of $\mathbb{N}^N$. If $f(., x)$ is a flow to an input profile $x \in \mathbb{N}^N$, the set $S(f, x) = \{z \in [0, x] | f(z, x) \neq 0\}$ is called the *support* of $f(., x)$. For all $k \in \mathbb{N}$, a flow $f(., ke^N)$ to ke^N is *diagonal* if $S(f, ke^N) \subseteq D^<$, and *nearly diagonal* if $S(f, ke^N) \subseteq D^\leq$. A fixed flow system f is (nearly) diagonal if $f(., ke^N)$ is (nearly) diagonal for all $k \in \mathbb{N}$.

A quick description of $D^<$ and $D^\leqq$ may be useful. For all $t \in \mathbb{N}$, define $\Delta(t) = \{z \in \mathbb{N}^N | \ z(N) = t\}$ and let $D^<(t) = D^< \cap \Delta(t)$ and $D^\leqq(t) = D^\leqq \cap \Delta(t)$.

If t is a multiple of n, say $t = rn$ for some $r \in \mathbb{N}$, then $D^<(t) = \{re^N\}$. If $t = rn + 1$ for some $r \in \mathbb{N}$, $D^<(t)$ is made up of all permutations of the point $(r+1, r, ..., r)$. More generally, if $t = rn + k$ for some $r \in \mathbb{N}$ and $k \in \{1, ..., n-1\}$, $D^<(t)$ is made up of all permutations of the point $(r+1)e^{\{1,...,k\}} + re^{\{k+1,...,n\}}$.

The set $D^\leqq(t)$ coincides with $D^<(t)$ whenever t is not a positive multiple of n. If t is a positive multiple of n, say $t = rn$ for some $r \in \mathbb{N}\backslash\{0\}$, then $D^\leqq(t)$ is larger than $D^<(t)$. It contains all points that can be written as $(r+1)e^{\{1,...k\}} + re^{\{k+1,...,n-k\}} + (r-1)e^{\{n-k+1,...,n\}}$ for some integer k, $0 \leq k \leq \frac{n}{2}$ (with the understanding that $e^\emptyset = 0$), and all their permutations.

Figures 1 and **2** illustrate $D^<$ and $D^\leqq$ when $n = 2$ and 3 respectively.

It is well known that there is only one anonymous flow to the one-unit-input profile e^N, namely

$$f_i^S(z, e^N) = \frac{n_i(z)!(n - n_i(z) - 1)!}{n!} \text{ for all } i \in N \text{ such that } z_i = 1,$$

where $n_i(z) = |\{j \in N\backslash i \mid z_j = 1\}|$. As an illustration, **Figure 3** depicts $f^S(., e^{\{1,2,3\}})$. On the subset $\{(F, e^N) \mid F \in \mathcal{F}(N)\}$ (that is, on the problems corresponding to standard cooperative games), the flow $f^S(., e^N)$ generates the Shapley value.

Next, fix $k \in \mathbb{N}$. Because the support of a diagonal flow to ke^N is included in the union of the unit cubes $[0, e^N]$, $[e^N, 2e^N]$, ..., $[(k-1)e^N, ke^N]$, there is a unique anonymous diagonal flow to ke^N : we denote it $f^s(., ke^N)$. **Figure 4** depicts $f^s(., 2e^{\{1,2,3\}})$. Recalling the remarks following Definition 2, the subsystem $\{f^s(., ke^N) | k \in \mathbb{N}\}$ completely determines a unique fixed flow system, which we denote f^s.

Definition 4. The *serial method* φ^s is the output-sharing method represented by the unique anonymous diagonal fixed flow system f^s.

This method was proposed by Moulin (1995). When all inputs are perfect substitutes, that is, when the production function takes the form $F(z) = \widetilde{F}(z(N))$ for some function $\widetilde{F}$ from $\mathbb{N}$ to $\mathbb{R}_+$, it reduces to the well-known serial mechanism (Moulin and Shenker, 1992): assuming without loss of generality that $x_1 \leq x_2 \leq ... \leq x_n$, the output shares are $\varphi_1(F, x) = \frac{1}{n}\widetilde{F}(nx_1)$,

$\varphi_2(F,x) = \frac{1}{n}\widetilde{F}(nx_1) + \frac{1}{n-1}[\widetilde{F}(x_1 + (n-1)x_2) - \widetilde{F}(nx_1)]$, ..., $\varphi_n(F,x) = \frac{1}{n}\widetilde{F}(nx_1) + \frac{1}{n-1}[\widetilde{F}(x_1 + (n-1)x_2) - \widetilde{F}(nx_1)] + ... + [\widetilde{F}(x_1 + x_2 + ... + x_n) - \widetilde{F}(x_1 + x_2 + ... + x_{n-2} + 2x_{n-1})]$.

The methods we will characterize are only slight variations of the serial method.

Definition 5. A *nearly serial* method is an output-sharing method represented by an anonymous nearly diagonal fixed flow system.

While there are several anonymous nearly diagonal fixed flow systems, they form a rather small family. Each such system f is conveniently represented by a single *anonymous nearly diagonal unbounded flow*, that is, a mapping $\overline{f} : \mathbb{N}^N \to \mathbb{R}^N_+$ such that $\overline{f}(z) = 0$ if $z \notin D^{\leq}$, $\sum_{i\in N} \overline{f}_i(e^i) = 1$,

$$\sum_{i\in N} \overline{f}_i(z) = \sum_{i\in N} \overline{f}_i(z + e^i) \text{ for all } z \in D^{\leq}\backslash\{0\}, \tag{9}$$

and

$$\overline{f}(\pi z) = \pi \overline{f}(z) \text{ for all } z \in D^{\leq}. \tag{10}$$

For all $x \in \mathbb{R}^N$, $f(.,x) = p_{[0,x]}\overline{f}$.

An anonymous nearly diagonal unbounded flow $\overline{f}$ is a relatively simple object. Defining $S(\overline{f}) = \{z \in \mathbb{N}^N |\ \overline{f}(z) \neq 0\}$, we have $S(\overline{f}) \subseteq D^{\leq}$, that is, $S(\overline{f}) \cap \Delta(t) \subseteq D^{\leq}(t)$ for all $t \in \mathbb{N}$.

If t is not a multiple of n, recall that $D^{\leq}(t) = D^{<}(t)$, which is a *perfectly symmetric* set: each element of $D^{<}(t)$ is a permutation of *any* other element of $D^{<}(t)$ (see **Figure 2** for an illustration). In that case the restriction $S(\overline{f}) \cap \Delta(t) \subseteq D^{\leq}(t)$ and the anonymity condition (10) imply that

$$S(\overline{f}) \cap \Delta(t) = D^{<}(t)$$

and therefore $\overline{f}$ is completely determined on $\Delta(t)$ once it is determined on $\Delta(t-1)$.

When t *is* a positive multiple of n, then $D^{\leq}(t)$ is a strict superset of $D^{<}(t)$ but $S(\overline{f}) \cap \Delta(t)$ is necessarily a rather small subset of $D^{\leq}(t)$. In fact, for all $r \in \mathbb{N}\backslash\{0\}$ and $z \in S(\overline{f}) \cap \Delta(rn)$,

$$z = re^N \text{ or } z \text{ is a permutation of } (r+1)e^1 + re^{\{2,\ldots,n-1\}} + (r-1)e^n,$$

that is, $S(\overline{f}) \cap \Delta(rn)$ is included in the union of a singleton and a perfectly symmetric set. This follows from the flow conservation constraints

(9) combined with the fact that $S(\overline{f}) \subseteq D^{\leq}$. For instance, if $n = 4$, then $(1,1,3,3) \in D^{\leq}(8)$ but $(1,1,3,3) \notin S(\overline{f})$ because $(0,1,3,3)$, $(1,0,3,3)$, $(1,1,2,3)$, $(1,1,3,2) \notin D^{\leq}(7)$. (Note that $S(\overline{f}) \cap \Delta(rn)$ is necessarily a strict subset of $D^{\leq}(rn)$ only when $n \geq 4$.)

These observations imply that $\overline{f}$ is fully characterized by a sequence $\{\alpha_r\}_{r \in \mathbb{N} \backslash \{0\}}$ in $[0,1]$, where

$$\alpha_r = \sum_{i \in N} \overline{f}_i(re^N) \tag{11}$$

is the fraction of the total (unit) flow that goes through re^N. This is illustrated in **Figure 5** for $n = 2$ and in **Figures 6.a and 6.b** for $n = 3$. Choosing $\alpha_r = 1$ for all r guarantees that $\overline{f}$ is in fact diagonal, that is, $S(\overline{f}) \subseteq D^{<}$, and generates the serial method.

We now come to our central axiom and our main result. We ask that when all agents in a group increase their input, not all of them get less output.

Group Monotonicity. For all $F \in \mathcal{F}(N)$, all $x, x' \in \mathbb{N}^N$, and all nonempty $S \subseteq N$, $\{x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \backslash S\} \Rightarrow \{$there exists $i \in S$ such that $\varphi_i(F,x) \leq \varphi_i(F,x')\}$.

This property is a natural generalization of the standard Monotonicity axiom, which corresponds to the case where S is a singleton. As discussed in the Introduction, Group Monotonicity is best defended from the strategic angle. In a context where agents can easily coordinate their actions, the condition is necessary to prevent joint decreases in inputs. It is also sufficient if monetary transfers between agents can be credibly forbidden. Otherwise, a stronger condition becomes necessary: a coordinated reduction of inputs by a group of agents should never lead to a bigger *aggregate* output share for that group. This *Strong Group Monotonicity* condition is studied in Moulin and Sprumont (2005) who show in their Proposition 2 that it is incompatible with the combination of Additivity and Dummy if there are at least three agents.

Moulin and Sprumont (2005) prove that the serial method satisfies Group Monotonicity. In fact, it passes the test "with a slack": if the agents in S all decrease their input, one checks that *all agents with minimal input in* S get an output share smaller than or equal to their original share. This suggests that more methods could be group-monotonic. As it turns out, the other nearly serial methods also satisfy Group Monotonicity –but "barely": it may happen that only *some* agent with minimal input in S suffers an output share reduction. No other method satisfying our other conditions passes the test.

Theorem 2. *Let N be a finite set containing at least three agents. An output-sharing method φ for N satisfies Additivity, Zero Output for Zero Input, Irrelevance of Dummy Changes, Anonymity, and Group Monotonicity if and only if φ is a nearly serial method.*

Proof. Let $N = \{1, 2, ..., n\}$ be a set of agents, $n \geq 3$.

"If". Let φ be a nearly serial method for N. The only statement requiring a proof is that φ satisfies Group Monotonicity. Let $\overline{f}$ be the unbounded flow representing φ. Since it is anonymous and nearly diagonal, $\overline{f}$ is fully characterized by a sequence of weights $\{\alpha_r\}$ as in (11). Let $F \in \mathcal{F}(N)$, $S \subseteq N$, and let $x, x' \in \mathbb{N}^N$ be such that $x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \backslash S$. Since all fixed-flow methods satisfy Monotonicity (see Moulin and Sprumont, 2005), we may assume that S contains at least two agents. Because of the budget balance condition, we may also assume $S \neq N$. To shorten notation, write $f(.,x) = g$ and $f(.,x') = g'$. These flows obtain by projecting $\overline{f}$ on $[0,x]$ and $[0,x']$, respectively:

$$g = p_{[0,x]}\overline{f} \tag{12}$$

and

$$g' = p_{[0,x']}\overline{f}. \tag{13}$$

For ease of exposition, we give the argument for the case of three agents first.

Proof for $n = 3$.

Without loss of generality, let $S = \{1, 2\}$. Thus $x_1 < x'_1$, $x_2 < x'_2$ and $x_3 = x'_3$. Define $\overline{f}_{12} : \mathbb{N}^{\{1,2,3\}} \to \mathbb{R}_+^{\{1,2\}}$ by $\overline{f}_{12}(z) = (\overline{f}_1(z), \overline{f}_2(z))$. Similarly, define $g_{12} = (g_1, g_2)$ on $[0,x]$ and $g'_{12} = (g'_1, g'_2)$ on $[0,x']$.

Case 1. $x_1 \neq x_2$.

Suppose that $x_1 < x_2$. Because $\overline{f}$ is nearly diagonal, (12) and (13) give

$$g_1(z) = g'_1(z) \text{ for all } z \in [0,x].$$

It follows that $\varphi_1(F,x) \leq \varphi_1(F,x')$. Permuting the roles of 1 and 2, the same argument shows that $\varphi_2(F,x) \leq \varphi_2(F,x')$ when $x_2 < x_1$.

Case 2. $x_1 = x_2$.

For $r = 0, 1, ..., x_3$, define $X(r) = \{z \in [0, x] \mid z_3 = r\}$ and $X'(r) = \{z \in [0, x'] \mid z_3 = r\}$. For $i = 1, 2$, write $\varphi_i(F, x) = \sum_{r=0}^{x_3} y_i(r)$ and $\varphi_i(F, x') = \sum_{r=0}^{x_3} y_i'(r)$, where

$$y_i(r) = \sum_{z \in X(r)} g_i(z) \partial_i F(z), \tag{14}$$

and

$$y_i'(r) = \sum_{z \in X'(r)} g_i'(z) \partial_i F(z). \tag{15}$$

In order to compare $\varphi_i(F, x)$ with $\varphi_i(F, x')$, we will compare $y_i(r)$ with $y_i'(r)$ for all $r = 0, 1, ..., x_3$. This requires computing the mappings $g_{12}|_{X(r)}$ and $g_{12}'|_{X'(r)}$ for all r.

If $r < x_3$, these mappings are obtained by projecting respectively on $X(r)$ and $X'(r)$ the restriction of $\overline{f}_{12}$ to $Z(r) = \{z \in \mathbb{N}^{\{1,2,3\}} \mid z_3 = r\}$. Formally,

$$g_{12}|_{X(r)} = p_{X(r)}(\overline{f}_{12}|_{Z(r)}) \text{ and } g_{12}'|_{X'(r)} = p_{X'(r)}(\overline{f}_{12}|_{Z(r)}). \tag{16}$$

For all $r \in \mathbb{N} \backslash \{0\}$, the mapping $\overline{f}_{12}|_{Z(r)}$ in (16) is fully determined by the three numbers α_{r-1}, α_r, α_{r+1}:

$$\begin{aligned}
\overline{f}_1(r, r, r) &= \overline{f}_1(r+1, r, r) = \frac{\alpha_r}{3}, \\
\overline{f}_1(r, r-1, r) &= \overline{f}_1(r+1, r+1, r) = \frac{1}{6}, \\
\overline{f}_1(r-1, r-1, r) &= \frac{1-\alpha_{r-1}}{6}, \\
\overline{f}_1(r+1, r-1, r) &= \overline{f}_1(r, r+1, r) = \frac{1-\alpha_r}{6}, \\
\overline{f}_1(r+2, r+1, r) &= \frac{1-\alpha_{r+1}}{6}, \\
\overline{f}_1(., ., r) &= 0 \text{ otherwise},
\end{aligned}$$

with the convention that $\alpha_0 = 1$, and $\overline{f}_2(., ., r)$ obtains by permutation. See **Figure 7**. The restriction of $\overline{f}_{12}$ to $Z(0)$ is simply $\overline{f}_1(1, 0, 0) = \frac{1}{3}$, $\overline{f}_1(1, 1, 0) = \frac{1}{6}$, $\overline{f}_1(2, 1, 0) = \frac{1-\alpha_1}{6}$, $\overline{f}_1(., ., 0) = 0$ otherwise, and $\overline{f}_2(., ., 0)$ obtains by permutation.

If $r = x_3$, the mappings $g_{12}|_{X(x_3)}$ and $g_{12}'|_{X'(x_3)}$ are obtained by projecting on $X(x_3)$ and $X'(x_3)$ the first two components of the unbounded flow $\overline{f}$ in or above $Z(x_3)$: letting $Z^{\geq}(x_3) = \{z \in \mathbb{N}^{\{1,2,3\}} \mid z_3 \geq x_3\}$, we have $g_{12}|_{X(x_3)} =$

$p_{X(x_3)}(\overline{f}_{12}|_{Z^{\geq}(x_3)})$ and $g'_{12}|_{X'(x_3)} = p_{X'(x_3)}(\overline{f}_{12}|_{Z^{\geq}(x_3)})$. It is convenient to compute these mappings in two steps. First we compute $p_{Z(x_3)}(\overline{f}_{12}|_{Z^{\geq}(x_3)})$. Denoting this mapping h_{12}, we find

$$\begin{aligned}
h_1(x_3-1, x_3-1, x_3) &= \frac{1-\alpha_{x_3-1}}{6}, \\
h_1(x_3, x_3, x_3) &= \frac{1+\alpha_{x_3}}{6}, \\
h_1(x_3+m, x_3+m, x_3) &= h_1(x_3+m+1, x_3+m, x_3) \\
&= \frac{2+\alpha_{x_3+m}}{6} \text{ for all } m \in \mathbb{N}\backslash\{0\}, \\
h_1(x_3, x_3+1, x_3) &= \frac{1-\alpha_{x_3}}{6},
\end{aligned}$$

$$\begin{aligned}
h_1(x_3+m, x_3+m+1, x_3) &= \frac{2-\alpha_{x_3+m}-\alpha_{x_3+m+1}}{6} \text{ for all } m \in \mathbb{N}\backslash\{0\}, \\
h_1(x_3, x_3-1, x_3) &= \frac{1}{6}, \\
h_1(x_3+1, x_3, x_3) &= \frac{1+2\alpha_{x_3}}{6}, \\
h_1(x_3+m+1, x_3+m-1, x_3) &= \frac{1-\alpha_{x_3+m}}{6}, \text{ all } m \in \mathbb{N}, \\
h_1(.,., x_3) &= 0 \text{ otherwise},
\end{aligned}$$

and we obtain h_2 by permutation. The mapping h_{12} is illustrated on **Figure 8**. The second step consists in projecting this mapping on $X(x_3)$ and $X'(x_3)$ to obtain

$$g_{12}|_{X(x_3)} = p_{X(x_3)}h_{12} \text{ and } g'_{12}|_{X'(x_3)} = p_{X'(x_3)}h_{12}. \tag{17}$$

We are now ready to compare $y_i(r)$ with $y'_i(r)$ for $i \in \{1,2\}$ and all possible values of r. We make two claims. In both claims, $r \in \{0, ..., x_3\}$ and ρ denotes the common value of x_1 and x_2.

Claim 1. *If* $r \neq \min\{\rho, x_3\}$, *then* $y_i(r) \leq y'_i(r)$ *for all* $i \in \{1,2\}$.

There are only two possibilities.

1.1. $r < x_3$ and $r \neq \rho$.

The mappings $g_{12}|_{X(r)}$ and $g'_{12}|_{X'(r)}$ are given by (16), with the mapping $\overline{f}_{12}|_{Z(r)}$ shown on **Figure 7**. Recall that $X(r) = \{z \in [0,x] \mid z_3 = r\}$, where

$x = (\rho, \rho, x_3)$. Because $r \neq \rho$, we have either $\rho \leq r-1$ or $\rho \geq r+1$. In either case, we find that g_{12} and g'_{12} coincide on $X(r)$, namely, $g_{12}(z) = g'_{12}(z) = \overline{f}_{12}(z)$ for all $z \in X(r)$. The claim follows from (14) and (15).

1.2. $\rho < r = x_3$.

The mappings $g_{12}|_{X(r)} = g_{12}|_{X(x_3)}$ and $g'_{12}|_{X'(r)} = g'_{12}|_{X'(x_3)}$ are now given by (17) with the mapping h_{12} illustrated on **Figure 8**. Because $\rho < x_3$, these mappings coincide on $X(x_3)$, that is, $g_{12}(z) = g'_{12}(z)$ for all $z \in X(x_3)$, and the claim follows again from (14) and (15).

Claim 2. *If* $r = \min\{\rho, x_3\}$, *then* $y_i(r) \leq y'_i(r)$ *for some* $i \in \{1,2\}$.

There are only two possibilities.

2.1. $r = \rho < x_3$.

The mappings $g_{12}|_{X(r)} = g_{12}|_{X(\rho)}$ and $g'_{12}|_{X'(r)} = g'_{12}|_{X'(\rho)}$ are given by (16) with $r = \rho$. They are illustrated in **Figures 9.a** and **9.b**. Unless $\alpha_\rho = 1$, these mappings do not coincide on $X(r) = X(\rho) = \{z \in [0,x] \mid z_3 = \rho\}$ since $g_{12}(\rho,\rho,\rho) = (\frac{1+\alpha_\rho}{6}, \frac{1+\alpha_\rho}{6}) \neq (\frac{\alpha_\rho}{3}, \frac{\alpha_\rho}{3}) = g'_{12}(\rho,\rho,\rho)$. However, we show that $y_i(\rho) \leq y'_i(\rho)$ for *some* $i \in \{1,2\}$.

We compute

$$y_1(\rho) = \frac{1-\alpha_{\rho-1}}{6}\partial_1 F(\rho-1,\rho-1,\rho) + \frac{1}{6}\partial_1 F(\rho,\rho-1,\rho) + \frac{1+\alpha_\rho}{6}\partial_1 F(\rho,\rho,\rho),$$

$$\begin{aligned} y'_1(\rho) \;\; \geq \;\; & \frac{1-\alpha_{\rho-1}}{6}\partial_1 F(\rho-1,\rho-1,\rho) + \frac{1}{6}\partial_1 F(\rho,\rho-1,\rho) \\ & + \frac{\alpha_\rho}{3}\partial_1 F(\rho,\rho,\rho) + \frac{1-\alpha_\rho}{6}\partial_1 F(\rho,\rho+1,\rho) + \frac{1-\alpha_\rho}{6}\partial_1 F(\rho+1,\rho-1,\rho), \end{aligned}$$

and obtain corresponding expressions for $y_2(\rho)$ and $y'_2(\rho)$ by permuting the first and second argument of the production function.

Taking differences,

$$\begin{aligned} y'_1(\rho) - y_1(\rho) \;\; & \geq \;\; \frac{1-\alpha_\rho}{6}\left(\partial_1 F(\rho,\rho+1,\rho) + \partial_1 F(\rho+1,\rho-1,\rho) - \partial_1 F(\rho,\rho,\rho)\right), \\ y'_2(\rho) - y_2(\rho) \;\; & \geq \;\; \frac{1-\alpha_\rho}{6}\left(\partial_2 F(\rho+1,\rho,\rho) + \partial_2 F(\rho-1,\rho+1,\rho) - \partial_2 F(\rho,\rho,\rho)\right). \end{aligned}$$

Suppose now, contrary to the claim, that $y'_i(\rho) < y_i(\rho)$ for $i = 1,2$. Then

$$\begin{aligned} \partial_1 F(\rho,\rho,\rho) \;\; & > \;\; \partial_1 F(\rho,\rho+1,\rho) + \partial_1 F(\rho+1,\rho-1,\rho) \text{ and} \\ \partial_2 F(\rho,\rho,\rho) \;\; & > \;\; \partial_2 F(\rho+1,\rho,\rho) + \partial_2 F(\rho-1,\rho+1,\rho). \end{aligned}$$

But since F is nondecreasing,

$$\begin{aligned}\partial_1 F(\rho,\rho,\rho) &\leq \partial_1 F(\rho,\rho+1,\rho)+\partial_2 F(\rho-1,\rho+1,\rho) \text{ and}\\ \partial_2 F(\rho,\rho,\rho) &\leq \partial_2 F(\rho+1,\rho,\rho)+\partial_1 F(\rho+1,\rho-1,\rho),\end{aligned}$$

which are incompatible with the two strict inequalities just derived.

2.2. $r = x_3 \leq \rho$.

The mappings $g_{12}|_{X(r)} = g_{12}|_{X(x_3)}$ and $g'_{12}|_{X'(r)} = g'_{12}|_{X'(x_3)}$ are now given by (17). Because $x_3 \leq \rho$, these mappings need not coincide on $X(x_3)$ but the argument in **2.1** can be mimicked with $g_{12}|_{X(x_3)}$ and $g'_{12}|_{X'(x_3)}$ replacing $g_{12}|_{X(\rho)}$ and $g'_{12}|_{X'(\rho)}$, respectively.

Claims 1 and 2 together establish that $\varphi_i(F,x) \leq \varphi_i(F,x')$ for some $i \in \{1,2\}$ when $x_1 = x_2 = \rho$, completing the proof of Group Monotonicity when $n = 3$.

Proof for any $n \geq 3$.

Define $T = \{i \in S \mid x_i \leq x_j \text{ for all } j \in S\}$. Since $S \neq N$, $N \backslash T \neq \emptyset$.

Case 1. T is a singleton, say $T = \{i\}$.

Because $\overline{f}$ is nearly diagonal, (12) and (13) give

$$g_i(z) = g'_i(z) \text{ for all } z \in [0,x]$$

and it follows that $\varphi_i(F,x) \leq \varphi_i(F,x')$.

Case 2. T is not a singleton.

By definition, all agents in T contribute the same number of units, say, $x_i = \rho$ for all $i \in T$. Define $\overline{f}_T : \mathbb{N}^N \to \mathbb{R}^T_+$ by $\overline{f}_T(z) = (\overline{f}_i(z))_{i\in T}$. Similarly, define $g_T = (g_i)_{i\in T}$ on $[0,x]$ and $g'_T = (g'_i)_{i\in T}$ on $[0,x']$. For all $r_{N\backslash T} \in [0,x_{N\backslash T}]$, define $X(r_{N\backslash T}) = \{z \in [0,x] \mid z_{N\backslash T} = r_{N\backslash T}\}$ and $X'(r_{N\backslash T}) = \{z \in [0,x'] \mid z_{N\backslash T} = r_{N\backslash T}\}$. For all $i \in T$, write $\varphi_i(F,x) = \sum_{r_{N\backslash T}\in[0,x_{N\backslash T}]} y_i(r_{N\backslash T})$ and $\varphi_i(F,x') = \sum_{r_{N\backslash T}\in[0,x'_{N\backslash T}]} y'_i(r_{N\backslash T}) \geq \sum_{r_{N\backslash T}\in[0,x_{N\backslash T}]} y'_i(r_{N\backslash T})$, where

$$y_i(r_{N\backslash T}) = \sum_{z\in X(r_{N\backslash T})} g_i(z)\partial_i F(z)$$

and

$$y'_i(r_{N\backslash T}) = \sum_{z\in X'(r_{N\backslash T})} g'_i(z)\partial_i F(z).$$

In order to compare $\varphi_i(F,x)$ with $\varphi_i(F,x')$, we will compare $y_i(r_{N\setminus T})$ with $y_i'(r_{N\setminus T})$ for all $r_{N\setminus T}\in[0,x_{N\setminus T}]$.

Fix $r_{N\setminus T}\in[0,x_{N\setminus T}]$ and let $Z(r_{N\setminus T})=\{z\in\mathbb{N}^N|\ z_{N\setminus T}=r_{N\setminus T}\}$. Let $M^<=\{i\in N\setminus T|\ r_i<x_i\}$, $M^==\{i\in N\setminus T|\ r_i=x_i\}$, and let $Z^{\geq}(r_{N\setminus T})=\{z\in\mathbb{N}^N|\ z_{M^<}=r_{M^<}$ and $z_{M^=}\geq x_{M^=}\}$. Define $h_T:Z(r_{N\setminus T})\to\mathbb{R}_+^T$ by

$$h_T=p_{Z(r_{N\setminus T})}(\overline{f}_T|_{Z^{\geq}(r_{N\setminus T})}).$$

By construction, this mapping is nearly diagonal with respect to T: if $(z_T,r_{N\setminus T})\in Z(r_{N\setminus T})$, then $h_T(z_T,r_{N\setminus T})=0$ whenever $z_T\notin D_T^{\lessapprox}=\{z_T\in\mathbb{N}^T|\ |z_i-\frac{z(T)}{|T|}|\leq 1$ for all $i\in T\}$, the near diagonal of $\mathbb{N}^T$. Moreover, h_T is anonymous with respect to T: for all $(z_T,r_{N\setminus T})\in Z(r_{N\setminus T})$ and any permutation σ on T, $h_T(\sigma z_T,r_{N\setminus T})=\sigma h_T(z_T,r_{N\setminus T})$. Finally, by (12) and (13),

$$g_T|_{X(r_{N\setminus T})}=p_{X(r_{N\setminus T})}h_T \tag{18}$$

and

$$g_T'|_{X'(r_{N\setminus T})}=p_{X'(r_{N\setminus T})}h_T. \tag{19}$$

We are now ready to compare $y_i(r_{N\setminus T})$ with $y_i'(r_{N\setminus T})$ for $i\in T$. In the remainder of the proof, $r_{N\setminus T}\in[0,x_{N\setminus T}]$ and $z_{N\setminus T}\wedge z'_{N\setminus T}$ is defined for all $z,z'\in\mathbb{N}^N$ by $(z_{N\setminus T}\wedge z'_{N\setminus T})_i=\min\{z_i,z_i'\}$ for all $i\in N\setminus T$. We make two claims.

Claim 1. *If $r_{N\setminus T}\neq\rho e^N_{N\setminus T}\wedge x_{N\setminus T}$, then $y_i(r_{N\setminus T})\leq y_i'(r_{N\setminus T})$ for all $i\in T$.*

Fix $r_{N\setminus T}\neq\rho e^N_{N\setminus T}\wedge x_{N\setminus T}$. It is enough to show that

$$g_T(z)=g_T'(z)\text{ for all }z\in X(r_{N\setminus T}). \tag{20}$$

Indeed, it then follows that for all $i\in T$, $y_i'(r_{N\setminus T})=\sum_{z\in X'(r_{N\setminus T})}g_i'(z)\partial_iF(z)$ $\geq\sum_{z\in X(r_{N\setminus T})}g_i'(z)\partial_iF(z)=\sum_{z\in X(r_{N\setminus T})}g_i(z)\partial_iF(z)=y_i(r_{N\setminus T})$.

We distinguish two cases.

1.1. For all $i\in N\setminus T$, $r_i<x_i$.

In this case, the assumption $r_{N\setminus T}\neq\rho e^N_{N\setminus T}\wedge x_{N\setminus T}$ implies

$$r_i\neq\rho\text{ for some }i\in N\setminus T. \tag{21}$$

Because $M^{<} = N\backslash T$ and $M^{=} = \emptyset$, we have $Z^{\geq}(r_{N\backslash T}) = Z(r_{N\backslash T})$ and $h_T = p_{Z(r_{N\backslash T})}(\overline{f}_T|_{Z(r_{N\backslash T})}) = \overline{f}_T|_{Z(r_{N\backslash T})}$. Hence (18) and (19) become

$$g_T|_{X(r_{N\backslash T})} = p_{X(r_{N\backslash T})}(\overline{f}_T|_{Z(r_{N\backslash T})}) \tag{22}$$

and

$$g'_T|_{X'(r_{N\backslash T})} = p_{X'(r_{N\backslash T})}(\overline{f}_T|_{Z(r_{N\backslash T})}). \tag{23}$$

Without loss of generality, suppose $T = \{1, ..., t\}$ and $N\backslash T = \{t+1, ..., n\}$, $1 < t < n$. Because $\overline{f}$ is a nearly diagonal unbounded flow, we need only consider the following three cases:

(a) $r_{N\backslash T}$ is a permutation of $(k+1)e^{t+1}_{N\backslash T} + ke^{\{t+2,...,n-1\}}_{N\backslash T} + (k-1)e^{n}_{N\backslash T}$ for some $k \in \mathbb{N}\backslash\{0\}$ and with $t+2 \leq n-1$, say, $r_{N\backslash T} = (k+1, k, ..., k, k-1)$,
(b) $r_{N\backslash T}$ is a permutation of $(k+1)e^{\{t+1,...,t+m\}}_{N\backslash T} + ke^{\{t+m+1,...,n\}}_{N\backslash T}$ for some $k \in \mathbb{N}$ and $1 \leq m \leq n-t-1$, say, $r_{N\backslash T} = (k+1, ..., k+1, k, ..., k)$,
(c) $r_{N\backslash T} = ke^{\{t+1,...,n\}}$ for some $k \in \mathbb{N}$, which we write $r_{N\backslash T} = (k, ..., k)$.

If none of these cases prevails, then there is no z_T such that $(z_T, r_{N\backslash T}) \in S(\overline{f})$, that is, $\overline{f}(z) = 0$ for all $z \in Z(r_{N\backslash T})$, and (20) follows trivially from (22) and (23).

In case **(a)**, $(z_T, r_{N\backslash T}) \in S(\overline{f})$ only if $z_T = (k, ..., k)$. In such a case however, $\overline{f}_T(z_T, r_{N\backslash T}) = 0$. Indeed, if, say $\overline{f}_1(z_T, r_{N\backslash T}) \neq 0$, then $(z_T, r_{N\backslash T}) - e^1 = (k-1, k, ...k, k+1, k, ..., k, k-1) \in S(\overline{f})$, a contradiction to the fact that $\overline{f}$ is nearly diagonal. Thus $\overline{f}_T(z) = 0$ for all $z \in Z(r_{N\backslash T})$ and therefore (20) follows from (22) and (23).

In case **(b)**, $(z_T, r_{N\backslash T}) \in S(\overline{f})$ only if one of the following statements holds:
i) z_T is a permutation of $(k+2)e^1_T + (k+1)e^{\{2,...,t\}}_T$, say, $z_T = (k+2, k+1, ..., k+1)$,
ii) $z_T = (k+1)e^T_T$, which we write $z_T = (k+1, ..., k+1)$,
iii) z_T is a permutation of $(k+1)e^{\{1,...,p\}}_T + ke^{\{p+1,...,t\}}_T$where $1 \leq p < t$, say, $z_T = (k+1, ..., k+1, k, ..., k)$,
iv) $z_T = ke^T_T$, which we write $z_T = (k, ..., k)$,
v) z_T is a permutation of $(k-1)e^1_T + ke^{\{2,...,t\}}_T$, say, $z_T = (k-1, k, ..., k)$.

Observe that z_T always belongs not only to the near diagonal of $\mathbb{N}^T$, but to the diagonal of $\mathbb{N}^T$, $D^{<}_T = \{z_T \in \mathbb{N}^T | \; |z_i - \frac{z(T)}{|T|}| < 1$ for all $i \in T\}$. Since $X(r_{N\backslash T}) = \{z \in [0, x] \mid z_{N\backslash T} = r_{N\backslash T}\}$ and $x_T = \rho e^T_T$, it follows that

$p_{X(r_{N\backslash T})}(\overline{f}_T|_{Z(r_{N\backslash T})}) = \overline{f}_T|_{X(r_{N\backslash T})}$, that is to say, $g_T(z) = \overline{f}_T(z)$ for all $z \in X(r_{N\backslash T})$. This holds independently of whether $k < \rho$, $k = \rho$, or $k > \rho$. Moreover, $g'_T(z) = \overline{f}_T(z)$ for all $z \in X(r_{N\backslash T})$, proving (20).

In case **(c)**, $(z_T, r_{N\backslash T}) \in S(\overline{f})$ only if one of the following statements holds:
i) $z_T = (k+1)e_T^T$, which we write $z_T = (k+1, ..., k+1)$,
ii) z_T is a permutation of $(k+1)e_T^{\{1,...,p\}} + ke_T^{\{p+1,...,t\}}$where $1 \leq p < t$, say, $z_T = (k+1, ..., k+1, k, ...k)$,
iii) $z_T = ke_T^T$, which we write $z_T = (k, ..., k)$,
iv) z_T is a permutation of $ke_T^{\{1,...,p\}} + (k-1)e_T^{\{p+1,...,t\}}$where $1 \leq p < t$, say, $z_T = (k, ..., k, k-1, ..., k-1)$,
v) $z_T = (k-1)e_T^T$, which we write $z_T = (k-1, ..., k-1)$,
vi) z_T is a permutation of $(k+1)e_T^1 + ke_T^{\{2,...,t-1\}} + (k-1)e_T^t$ (with the convention $e_T^{\emptyset} = 0$), say, $z_T = (k+1, k, ..., k, k-1)$ (with the understanding that $z_T = (k+1, k-1)$ if $t = 2$).

Observe that z_T does not necessarily belong to the diagonal of $\mathbb{N}^T$ but $z_T \in \{k-1, k, k+1\}^T$. Now we use (21). Since $r_{N\backslash T} = (k, ..., k)$, it follows that $\rho \neq k$, that is, $\rho \leq k-1$ or $\rho \geq k+1$. Because of this, we have again $p_{X(r_{N\backslash T})}(\overline{f}_T|_{Z(r_{N\backslash T})}) = \overline{f}_T|_{X(r_{N\backslash T})}$, that is to say, $g_T(z) = \overline{f}_T(z)$ for all $z \in X(r_{N\backslash T})$. Likewise, $g'_T(z) = \overline{f}_T(z)$ for all $z \in X(r_{N\backslash T})$, and (20) follows.

1.2. There exists $i \in N\backslash T$ such that $r_i = x_i$.

In this case, $M^= \neq \varnothing$ and we assume without loss of generality that $T = \{1, ..., t\}$ and $M^= = \{m+1, ..., n\}$, where $t \leq m < n$. Also without loss, we suppose $x_i \leq x_n$ for all $i \in M^=$. Thus we have $r_{N\backslash T} = (r_{M^<}, x_{M^=})$ and the assumption $r_{N\backslash T} \neq \rho e_{N\backslash T}^N \wedge x_{N\backslash T}$ means that

$$\rho < x_n \text{ or there exists } i \in M^< \text{ such that } r_i \neq \rho. \tag{24}$$

We distinguish two subcases.

1.2.1. There exists $i \in M^<$ such that $r_i \neq \rho$.

The argument in this subcase mimics the argument in **1.1**. Again, we need only consider three cases:

(a) $r_{M^<}$ is a permutation of $(k+1, k, ..., k, k-1)$ for some $k \in \mathbb{N}\backslash\{0\}$,
(b) $r_{M^<}$ is a permutation of $(k+1, ..., k+1, k, ..., k)$ for some $k \in \mathbb{N}$,
(c) $r_{M^<} = (k, ..., k)$ for some $k \in \mathbb{N}$.

If none of these cases prevails, there does not exist $z_T \in \mathbb{N}^T$ and $z_{M^=} \geq x_{M^=}$ such that $\overline{f}(z_T, r_{M^<}, z_{M^=}) \neq 0$, hence $h_T(z) = 0$ for all $z \in Z(r_{M^<}, x_{M^=})$ and (20) follows from (22) and (23).

In case **(a)**, there exist $z_T \in \mathbb{N}^T$ and $z_{M^=} \geq x_{M^=}$ such that $\overline{f}(z_T, r_{M^<}, z_{M^=}) \neq 0$ only if $z_T = (k, ..., k)$. In this case however, $h_T(z_T, r_{M^<}, z_{M^=}) = 0$ and again (20) follows from (22) and (23).

In case **(b)**, there exist $z_T \in \mathbb{N}^T$ and $z_{M^=} \geq x_{M^=}$ such that $\overline{f}(z_T, r_{M^<}, z_{M^=}) \neq 0$ only if z_T belongs to $D_T^{\leq} = \{z_T \in \mathbb{N}^T | \ |z_i - \frac{z(T)}{|T|}| < 1 \text{ for all } i \in T\}$. Since $X(r_{M^<}, x_{M^=}) = \{z \in [0, x] \mid (z_{M^<}, z_{M^=}) = (r_{M^<}, x_{M^=})\}$ and $x_T = \rho e_T^T$, it follows that $p_{X(r_{M^<}, x_{M^=})} h_T = h_T|_{X(r_{M^<}, x_{M^=})}$, that is to say, $g_T(z) = h_T(z)$ for all $z \in X(r_{M^<}, x_{M^=})$. Moreover, $g'_T(z) = h_T(z)$ for all $z \in X(r_{M^<}, x_{M^=})$, proving (20).

In case **(c)**, there exist $z_T \in \mathbb{N}^T$ and $z_{M^=} \geq x_{M^=}$ such that $\overline{f}(z_T, r_{M^<}, z_{M^=}) \neq 0$ only if $z_T \in \{k-1, k, k+1\}^T$. Now we use the fact that $r_i \neq \rho$ for some $i \in M^<$. Since $r_{M^<} = (k, ..., k)$, it follows that $\rho \neq k$, that is, $\rho \leq k-1$ or $\rho \geq k+1$. Because of this, we have $p_{X(r_{M^<}, x_{M^=})} h_T = h_T|_{X(r_{M^<}, x_{M^=})}$, that is to say, $g_T(z) = h_T(z)$ for all $z \in X(r_{M^<}, x_{M^=})$. Moreover, $g'_T(z) = h_T(z)$ for all $z \in X(r_{M^<}, x_{M^=})$, proving (20).

1.2.2. $r_i = \rho$ for all $i \in M^<$.

In this case (24) implies that $\rho \leq x_n - 1$.

We claim that if $i \in T$ and $(z_T, r_{M^<}, x_{M^=}) \in Z(r_{M^<}, x_{M^=})$, then

$$h_i(z_T, r_{M^<}, x_{M^=}) \neq 0 \Rightarrow z_T = (x_n - 1)e_T^T \text{ or } z_T \geq (x_n - 1)e_T^T + e_T^i. \quad (25)$$

Let $pD^{\leq} = \{(z_T, r_{M^<}, x_{M^=}) | \exists z_{M^=} \geq x_{M^=} : (z_T, r_{M^<}, z_{M^=}) \in D^{\leq}\}$. Fix $i \in T$, $(z_T, r_{M^<}, x_{M^=}) \in Z(r_{M^<}, x_{M^=})$, and suppose $h_i(z_T, r_{M^<}, x_{M^=}) \neq 0$. By definition of h_T,

$$(z_T, r_{M^<}, x_{M^=}) \in pD^{\leq} \quad (26)$$

and

$$(z_T - e_T^j, r_{M^<}, x_{M^=}) \in pD^{\leq} \text{ for some } j \in T. \quad (27)$$

From (26) we have $z_T \geq (x_n - 2)e_T^T$. It follows that

$$z_T \geq (x_n - 1)e_T^T \quad (28)$$

because if, say, $z_1 = x_n - 2$, then (26) implies $z_j = x_n - 1$ for all $j \in T \backslash 1$, hence $(z_T - e_T^j, r_{M^<}, x_{M^=}) \notin pD^{\leq}$ for all $j \in T$, contradicting (27). Now assume, contrary to (25), that neither $z_T = (x_n - 1)e^T_T$ nor $z_T \geq (x_n - 1)e^T_T + e^i_T$. By (28), $z_T \geq (x_n - 1)e^T_T + e^j_T$ for some $j \in T \backslash i$. But then $(z_T - e^i_T, r_{M^<}, x_{M^=}) \notin pD^{\leq}$, implying $h_i(z_T, r_{M^<}, x_{M^=}) = 0$, a contradiction.

Because (25) holds for all $i \in T$ and $(z_T, r_{M^<}, x_{M^=}) \in Z(r_{M^<}, x_{M^=})$ and because $\rho \leq x_n - 1$, we have $p_{X(r_{M^<}, x_{M^=})} h_T = h_T|_{X(r_{M^<}, x_{M^=})}$. Therefore g_T and g'_T coincide on $X(r_{M^<}, x_{M^=})$, proving (20).

Claim 2. *If $r_{N\backslash T} = \rho e^N_{N\backslash T} \wedge x_{N\backslash T}$, then $y_i(\rho e^N_{N\backslash T} \wedge x_{N\backslash T}) \leq y'_i(\rho e^N_{N\backslash T} \wedge x_{N\backslash T})$ for some $i \in T$.*

2.1. Let us first prove this claim under the assumption that $\rho e^N_{N\backslash T} \ll x_{N\backslash T}$. The projection formulas (18) and (19) yield

$$\begin{aligned} & y'_i(\rho e^N_{N\backslash T}) - y_i(\rho e^N_{N\backslash T}) \\ \geq\ & \frac{1-\alpha_\rho}{n(n-1)} \sum_{j \in T\backslash i} [\partial_i F((\rho+1)e^i + (\rho-1)e^j + \rho e^{N\backslash ij}) \\ & + \partial_i F((\rho+1)e^j + \rho e^{N\backslash j}) - \partial_i F\left(\rho e^N\right)] \end{aligned}$$

where $n(n-1)$ is the number of permutations of the point $(\rho+1)e^1 + \rho e^{\{2,\ldots,n-1\}} + (\rho-1)e^n$.

Suppose, by contradiction, that $y'_i(\rho e^N_{N\backslash T}) < y_i(\rho e^N_{N\backslash T})$ for all $i \in T$. Then $\sum_{j \in T\backslash i} \partial_i F\left(\rho e^N\right) > \sum_{j \in T\backslash i} [\partial_i F((\rho+1)e^i + (\rho-1)e^j + \rho e^{N\backslash ij}) + \partial_i F((\rho+1)e^j + \rho e^{N\backslash j})]$ for all $i \in T$. Summing up these inequalities,

$$\begin{aligned} & \sum_{i \in T} \sum_{j \in T\backslash i} \partial_i F\left(\rho e^N\right) \\ >\ & \sum_{i \in T} \sum_{j \in T\backslash i} [\partial_i F((\rho+1)e^i + (\rho-1)e^j + \rho e^{N\backslash ij}) + \partial_i F((\rho+1)e^j + \rho e^{N\backslash j})] \\ =\ & \sum_{i \in T} \sum_{j \in T\backslash i} [\partial_j F((\rho-1)e^i + (\rho+1)e^j + \rho e^{N\backslash ij}) + \partial_i F((\rho+1)e^j + \rho e^{N\backslash j})]. \end{aligned}$$

But since F is nondecreasing,

$$\partial_i F\left(\rho e^N\right) \leq \partial_j F((\rho-1)e^i + (\rho+1)e^j + \rho e^{N\backslash ij}) + \partial_i F((\rho+1)e^j + \rho e^{N\backslash j})$$

for all $i \in T$, $j \in T\backslash i$, a contradiction.

2.2. Dispensing now with the assumption $\rho e^N_{N\backslash T} \ll x_{N\backslash T}$, let $r_{N\backslash T} = \rho e^N_{N\backslash T} \wedge x_{N\backslash T}$. Because h_T is nearly diagonal and anonymous with respect to

T, we obtain

$$\begin{aligned} & y_i'(r_{N\backslash T}) - y_i(r_{N\backslash T}) \\ \geq\ & A_\rho \sum_{j\in T\backslash i} [\partial_i F\left((\rho+1)e_T^i + (\rho-1)e_T^j + \rho e_T^{N\backslash ij}, r_{N\backslash T}\right) \\ & + \partial_i F\left((\rho+1)e_T^j + \rho e_T^{N\backslash j}, r_{N\backslash T}\right) - \partial_i F\left(\rho e_T^N, r_{N\backslash T}\right)], \end{aligned}$$

where A_ρ is a coefficient that does not depend on i. If $y_i'(r_{N\backslash T}) < y_i(r_{N\backslash T})$ for all $i \in T$, a contradiction is obtained just as before.

"Only if". Let φ be an output-sharing method for N meeting all five axioms in Theorem 2. Recalling Theorem 1 and Lemma 2, φ is represented by a unique anonymous fixed flow system f. We will prove by induction on n that f is nearly diagonal.

Step 1. *f is nearly diagonal if $n = 3$.*

We fix $N = \{1,2,3\}$ and claim that $S(f, ke^N) \subseteq D^{\leq}$ for all $k \in \mathbb{N}\backslash\{0\}$. To prove our claim it suffices to show

$$\forall k \in \mathbb{N}\backslash\{0\},\ \forall t \in \mathbb{N}\backslash\{0\} \text{ such that } t \leq k,\ S(f, ke^N) \cap \Delta(t) \subseteq D^{\leq}(t). \quad (29)$$

(Indeed, let $k \in \mathbb{N}\backslash\{0\}$ and suppose $z \in S(f, ke^N)$. Set $t := z(N)$. If $t \leq k$, (29) implies $z \in D^{\leq}$, as desired. If $t > k$, simply choose $k' \geq t > k$. By the fixed-flow property, $z \in S(f, k'e^N)$, so that (29) implies $S(f, k'e^N) \cap \Delta(t) \subseteq D^{\leq}(t)$, hence $z \in D^{\leq}$ again.)

From now on, we fix $k \in \mathbb{N}\backslash\{0\}$ and (with a slight abuse of notation) write f instead of $f(., ke^N)$. We show by induction on t that $S(f) \cap \Delta(t) \subseteq D^{\leq}(t)$ for all $t \in \mathbb{N}\backslash\{0\}$.

If $t = 1$, the anonymity of f directly implies that

$$S(f) \cap \Delta(1) = D^{\leq}(1).$$

Next we fix t, $2 \leq t \leq k$, assume $S(f) \cap \Delta(\tau) \subseteq D^{\leq}(\tau)$ for $\tau = 1, ..., t-1$, and show that $S(f) \cap \Delta(t) \subseteq D^{\leq}(t)$. Given $z \in \mathbb{N}^N\backslash\{0\}$, we call $f(z) = (f_1(z), f_2(z), f_3(z))$ the flow at z. By the *flow at a set Z* we mean the collection $\{f(z) \mid z \in Z\}$. We distinguish three cases.

Case 1. $t = 3r + 1$ for some $r \in \mathbb{N}\backslash\{0\}$.

Then $D^{\leq}(t) = \{(r,r,r+1), (r,r+1,r), (r+1,r,r)\}$. By the induction hypothesis, $S(f) \cap \Delta(3r) \subseteq D^{\leq}(3r)$. The latter set is made up of 7 points:

(r,r,r) and the 6 permutations of $(r-1,r,r+1)$. The induction hypothesis also ensures that $S(f)\cap\Delta(3r-1)\subseteq D^{\leq}(3r-1)$, which is made up of the three permutations of $(r-1,r,r)$. By anonymity of f, the flow at $D^{\leq}(3r)$ is therefore determined up to one parameter $\alpha\in[0,1]$:

$$\begin{aligned} f(r,r,r) &= (\frac{\alpha}{3},\frac{\alpha}{3},\frac{\alpha}{3}),\\ f(r-1,r,r+1) &= (0,0,\frac{1-\alpha}{6}), \end{aligned}$$

and the flow at each permutation of $(r-1,r,r+1)$ obtains by applying that permutation to $f(r-1,r,r+1)$. See **Figure 10**.

We want to show that

$$S(f)\cap\Delta(3r+1)\subseteq D^{\leq}(3r+1),$$

where $D^{\leq}(3r+1)$ is made up of the three permutations of $(r,r,r+1)$. By the induction hypothesis, flow conservation and the anonymity of f, it suffices to show that

$$f_1(r+1,r+1,r-1)=0 \tag{30}$$

and

$$f_1(r+2,r,r-1)=0. \tag{31}$$

The flow at $D^{\leq}(3r+1)$ is then fully determined by flow conservation: $f(r+1,r,r)=(\frac{\alpha}{3},,\frac{1-\alpha}{6},\frac{1-\alpha}{6})$ and the flow at the permuted points obtains by permutation. See **Figure 11**.

Proving (30). Suppose, by way of contradiction, $f_1(r+1,r+1,r-1)=f_2(r+1,r+1,r-1)=\beta$, $0<\beta\leq\frac{1-\alpha}{6}$. Write $g=f(.,(r+1,r+1,r+1))$, the projection of f on $[0,(r+1)e^N]$, and note that g is fully determined. In particular, $g(r+1,r-1,r+1)=(\beta,0,\beta)$, $g(r-1,r+1,r+1)=(0,\beta,\beta)$, and $g(r,r,r+1)=(\frac{1-\alpha}{6}-\beta,\frac{1-\alpha}{6}-\beta,\frac{\alpha}{3})$. Define the production function F by $F(w)=1$ if $w\geq z$ for some $z\in\{(r,r,r+1),(r+1,r-1,r+1),(r-1,r+1,r+1)\}$ and $F(w)=0$ otherwise. Then $\varphi_1(F,(r+1,r+1,r+1))=\varphi_2(F,(r+1,r+1,r+1))=(\frac{1-\alpha}{6}-\beta)+\beta=\frac{1-\alpha}{6}$. But using the fixed-flow property, compute now $\varphi_1(F,(r,r,r+1))=g_1(r,r,r+1)+g_1(r,r+1,r+1)=(\frac{1-\alpha}{6}-\beta)+2\beta=\frac{1-\alpha}{6}+\beta$, which is also $\varphi_2(F,(r,r,r+1))$ by Anonymity. This violates Group Monotonicity. See **Figure 12**.

We have proved (30). In fact, we have shown more, namely $g(r+1,r+1,r-1)=(0,0,0)$. As a consequence, by flow conservation,

$$g_3(r+1,r+1,r)=0.$$

Proving (31). Suppose $f_1(r+2,r,r-1) = f_2(r,r+2,r-1) = \gamma$, $0 < \gamma \leq \frac{1-\alpha}{6}$. Define the production function F' by $F'(w) = 1$ if $w \geq z$ for some $z \in \{(r+1,r+1,r),(r+2,r,r),(r,r+2,r)\}$ and $F'(w) = 0$ otherwise. See **Figure 13.** Define

$$W = \left\{ \begin{array}{c} (r,r+2,r),(r+1,r+1,r),(r+2,r,r), \\ (r+1,r+2,r),(r+2,r+1,r),(r+2,r+2,r) \end{array} \right\}.$$

This set contains all the input profiles below $(r+2,r+2,r+1)$ where agent 3's marginal product is positive. In fact, $\partial_3 F'(w) = 1$ if $w \in W$ and $\partial_3 F'(w) = 0$ for all $w \in [0,(r+2,r+2,r+1)] \setminus W$.

By the representation formula (2), therefore, $\varphi_3(F',(r+1,r+1,r+1)) = g_3(r+1,r+1,r) = 0$.

Now let $g' = f(.,(r+2,r+2,r))$, the projection of f on $[0,(r+2,r+2,r)]$. By the fixed-flow property, $g'_1(r+2,r,r-1) = f_1(r+2,r,r-1) = \gamma$ and $g'_2(r,r+2,r-1) = f_2(r,r+2,r-1) = \gamma$. Moreover, $g'_3(r+1,r+1,r) = g_3(r+1,r+1,r) = 0$. By (2), $\varphi_3(F',(r+2,r+2,r)) = \sum_{w\in W} g'_3(w) = \sum_{w\in W\setminus(r+1,r+1,r)} g'_3(w)$. By flow conservation, $\sum_{w\in W\setminus(r+1,r+1,r)} g'_3(w) \geq g'_1(r+2,r,r-1) + g'_2(r,r+2,r-1) = 2\gamma > 0$, hence, $\varphi_3(F',(r+2,r+2,r)) > 0$. Since every fixed-flow method satisfies Monotonicity, $\varphi_3(F',(r+2,r+2,r+1)) \geq \varphi_3(F',(r+2,r+2,r)) > 0$.

Thus, $\varphi_3(F',(r+2,r+2,r+1)) > \varphi_3(F',(r+1,r+1,r+1))$. But $F'(r+2,r+2,r+1) = F'(r+1,r+1,r+1) = 1$. Using budget balance and Anonymity, it follows that $\varphi_i(F',(r+2,r+2,r+1)) < \varphi_i(F',(r+1,r+1,r+1))$ for $i = 1,2$, violating Group Monotonicity.

Case 2. $t = 3r+2$ for some $r \in \mathbb{N}$.

Then $D^{\leq}(t) = \{(r,r+1,r+1),(r+1,r,r+1),(r+1,r+1,r)\}$. By the induction hypothesis, $S(f) \cap \Delta(3r+1) \subseteq D^{\leq}(3r+1)$ and by anonymity of f, the flow at $D^{\leq}(3r+1)$ is as shown in **Figure 11**: $f(r+1,r,r) = (\frac{\alpha}{3},\frac{1-\alpha}{6},\frac{1-\alpha}{6})$ and the the flow at the permuted points obtains by permutation. We claim now that

$$S(f) \cap \Delta(3r+2) \subseteq D^{\leq}(3r+2). \tag{32}$$

Because of the anonymity of f, we need only show

$$f_1(r+2,r,r) = 0.$$

Suppose $f_1(r+2,r,r) = \delta$, $0 < \delta \leq \frac{1}{3}$. Define the production function F'' by $F''(w) = 1$ if $w \geq z$ for some $z \in \{(r+1,r+1,r+1),(r+2,r,r+1),(r,r+$

$2, r+1)\}$ and $F(w) = 0$ otherwise, as shown in **Figure 14**. By Anonymity, $\varphi_3(F'', (r+1, r+1, r+1)) = \frac{1}{3}$. On the other hand, by the fixed-flow property and flow conservation, $\varphi_3(F'', (r+2, r+2, r+1)) \geq \delta + (\frac{1}{6} - \frac{\delta}{2}) + (\frac{1}{6} - \frac{\delta}{2}) + \delta = \frac{1}{3} + \delta > \frac{1}{3}$. So $\varphi_i(F'', (r+1, r+1, r+1)) > \varphi_i(F'', (r+2, r+2, r+1))$ for $i = 1, 2$, contradicting Group Monotonicity. This proves (32). Note that the anonymity of f completely determines the entire flow at $D^{\leq}(3r+2)$: $f_1(r+1, r+1, r) = f_2(r+1, r+1, r) = f_1(r+1, r, r+1) = f_3(r+1, r, r+1) = f_2(r, r+1, r+1) = f_3(r, r+1, r+1) = \frac{1}{6}$, as depicted on **Figure 15**.

Case 3. $t = 3r$ for some $r \in \mathbb{N}\backslash\{0\}$.

Then $D^{\leq}(t) = \{(r, r, r), (r-1, r, r+1), (r-1, r+1, r), (r, r-1, r+1), (r, r+1, r-1), (r+1, r-1, r), (r+1, r, r-1)\}$. By the induction hypothesis, $S(f) \cap \Delta(3r-1) \subseteq D^{\leq}(3r-1)$ (and the flow at $D^{\leq}(3r-1)$ is as shown on **Figure 16**). It follows directly from flow conservation that $S(f) \cap \Delta(3r) \subseteq D^{\leq}(3r)$. Moreover, by anonymity of f, the flow at $D^{\leq}(3r)$ must be as shown on **Figure 10**: $f(r, r, r) = (\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3})$, $f(r-1, r, r+1) = (0, 0, \frac{1-\alpha}{6})$, and the flow at each permutation of $(r-1, r, r+1)$ obtains by applying that permutation to $f(r+1, r, r-1)$.

Step 2. *Induction argument.*

Let $N = \{1, ..., n\}$, $n > 3$. Make the induction hypothesis that for all $M \subseteq N$ such that $3 \leq |M| < n$, every output-sharing method for M satisfying our five axioms is a nearly serial method. Let φ be an output-sharing method for N satisfying the five axioms. Let f be the flow system representing φ. Because of Theorem 1 and Lemma 2, f is an anonymous fixed flow system. We must prove that it is nearly diagonal.

Suppose, by contradiction, that there exist $k > 0$, $z \in S(f, ke^N)$, and $i \in N$ such that

$$\left| z_i - \frac{z(N)}{n} \right| > 1. \tag{33}$$

We claim that

$$\left| z_i - \frac{z(N\backslash j)}{n-1} \right| > 1 \text{ for some } j \in N\backslash i. \tag{34}$$

To prove (34), decompose (33) into two cases: either $z_i - \frac{z(N)}{n} > 1$ or $\frac{z(N)}{n} - z_i > 1$. We only consider the former case, the latter is similar and left to the reader. Without loss of generality, assume $i = 1$, so that

$$(n-1)z_1 > z(N\backslash 1) + n. \tag{35}$$

Assume without loss of generality

$$z_1 \geq z_2 \geq ... \geq z_n. \tag{36}$$

In order to prove (34), it suffices to show that

$$(n-2)z_1 > z(N \backslash \{1,2\}) + (n-1).$$

Suppose, by contradiction, that

$$(n-2)z_1 \leq z(N \backslash \{1,2\}) + (n-1). \tag{37}$$

Then $(n-1)z_1 \leq z(N \backslash 2) + (n-1)$ and therefore, by (35), $z(N \backslash 1) + n < z(N \backslash 2) + (n-1)$. Hence,

$$z_2 + 1 < z_1. \tag{38}$$

By (36), $(z_3+1), ..., (z_n+1) < z_1$, hence $(n-2)z_1 > z(N \backslash \{1,2\}) + (n-2)$, which, combined with (37), gives $(n-2)z_1 = z(N \backslash \{1,2\}) + (n-1)$. Using (38),

$$z_2 < \frac{z(N \backslash \{1,2\}) + 1}{n-2}. \tag{39}$$

If at least one inequality is strict in the string $z_3 \geq z_4 \geq ... \geq z_n$, then $z_n + 1 \leq z_3$ and it follows that $z(N \backslash \{1,2\}) + 1 \leq (n-2)z_3$. Therefore (39) implies $z_2 < z_3$, a contradiction.

If $z_3 = z_4 = ... = z_n$, then (39) reads $z_2 < z_3 + \frac{1}{n-2}$ and, because of (36), $z_2 = z_3$. Thus (37) becomes $(n-2)z_1 \leq (n-2)z_2 + (n-1)$, hence $z_1 \leq z_2 + \frac{n-1}{n-2} < z_2 + 2$ and, since z_1, z_2 are integers, $z_1 \leq z_2 + 1$. This contradicts (38) and establishes (34).

Define the system $f^{-j} = \{f^{-j}(., x_{N \backslash j}) | \, x_{N \backslash j} \in \mathbb{N}^{N \backslash j} \backslash \{0\}\}$ by $f^{-j}(z_{N \backslash j}, x_{N \backslash j}) = f_{N \backslash j}((z_{N \backslash j}, 0_j), (x_{N \backslash j}, 0_j))$. Because f is an anonymous fixed flow system, f^{-j} is itself an anonymous fixed flow system for the agent set $N \backslash j$. The output-sharing method φ^{-j} for $N \backslash j$ that f^{-j} represents satisfies

$$\varphi^{-j}(F^{-j}, x_{N \backslash j}) = \varphi_{N \backslash j}(F, (x_{N \backslash j}, 0_j))$$

for all $x_{N \backslash j} \in \mathbb{N}^{N \backslash j}$ and $F \in \mathcal{F}(N)$, where $F^{-j}(z_{N \backslash j}) := F(z_{N \backslash j}, 0_j)$. Since φ satisfies Group Monotonicity, so does φ^{-j}. Because f is a fixed flow system, $z \in S(f, ke^N)$ implies $(z_{N \backslash j}, 0_j) \in S(f, ke^{N \backslash j})$. Hence, by definition of f^{-j},

$$z_{N \backslash j} \in S(f^{-j}, ke^{N \backslash j}), \tag{40}$$

and (34) and (40) contradict the induction hypothesis. ■

5 Discussion

We make four comments about Theorem 2.

1) It is easy to see that the axioms are independent. For a method satisfying all axioms but Additivity, consider the following adjusted proportional method. Given a problem (F, x), define $a^*_{iF} = \min\{a_i | \partial_i F(z) = 0$ for all z such that $z_i \geq a_i\}$ (with the convention that $a^*_{iF} = +\infty$ if the latter set is empty) and let $x^*_{iF} = \min\{x_i, a^*_{iF}\}$. Define $\varphi_i(F, x) = \frac{x^*_{iF}}{x^*_F(N)} F(x)$ if $x^*_F(N) > 0$ (and $\varphi(F, x) = 0$ if $x^*_F(N) = 0$). An example violating only Zero Output for Zero Input is the egalitarian method $\varphi_i(F, x) = \frac{1}{n} F(x)$. A method violating only Irrelevance of Dummy Changes is the proportional method $\varphi_i(F, x) = \frac{x_i}{x(N)} F(x)$ if $x(N) > 0$ (and $\varphi(F, 0) = 0$). For an example violating only Anonymity, consider any fixed-path method. Finally, the Shapley-Shubik method satisfies all axioms but Group Monotonicity. See Moulin and Sprumont (2005) for details.

2) The proof does not use the full force of Group Monotonicity. That axiom may be replaced in Theorem 2 by the following weaker requirement.

Pairwise Monotonicity. For all $F \in \mathcal{F}(N)$, all $x, x' \in \mathbb{N}^N$, and all pairs of distinct agents $i, j \in N$, $\{x_i < x'_i,\ x_j < x'_j$, and $x_k = x'_k$ for all $k \in N \backslash ij\} \Rightarrow \{\varphi_i(F, x) \leq \varphi_i(F, x')$ or $\varphi_j(F, x) \leq \varphi_j(F, x')\}$.

Observe that this condition does not imply Monotonicity. This is obvious for $n = 2$ since Pairwise Monotonicity is then implied by budget balance and therefore automatically satisfied by all output-sharing methods. For $n \geq 3$, consider the following method φ. For all $F \in \mathcal{F}(N)$ and $x_{-12} \in \mathbb{N}^{N \backslash \{1,2\}}$, define $F_{x_{-12}} : \mathbb{N}^{\{1,2\}} \to \mathbb{R}_+$ by $F_{x_{-12}}(z_1, z_2) = F(z_1, z_2, x_{-12}) - F(0, 0, x_{-12})$. Clearly, $F_{x_{-12}} \in \mathcal{F}(\{1, 2\})$. For all $(F, x) \in \mathcal{F}(N) \times \mathbb{N}^N$, let $\varphi_i(F, x) = 0$ for $i \in N \backslash \{1, 2\}$ and $\varphi_i(F, x) = \frac{1}{2} F(0, 0, x_{-12}) + \varphi_i^{AS}(F_{x_{-12}}, (x_1, x_2))$ for $i \in N \backslash \{1, 2\}$, where φ^{AS} denotes the Aumann-Shapley method (see Sprumont, 2005 for a formal definition in the discrete framework). The method φ meets Pairwise Monotonicity because agents 1 and 2 always share the entire output and the others always get zero. To see that φ violates Monotonicity, consider a profile where $x_{-12} = 0$ and use any example showing that the (two-agent) Aumann-Shapley method is not monotonic (see, for instance, Moulin, 1995).

3) We have assumed that the domain of a sharing method is $\mathcal{F}(N) \times \mathbb{N}^N$, where $\mathcal{F}(N)$ contains all nondecreasing real-valued functions on $\mathbb{N}^N$ satisfying $F(0) = 0$. On smaller domains, the nearly serial methods need not

be the only methods satisfying the five axioms in Theorem 2. For instance, the Shapley-Shubik method is group monotonic if we restrict our attention to production functions F that are submodular (in the sense that $\partial_i F(z + e^j) \leq \partial_i F(z)$ for all z and distinct i, j). We have not identified interesting subdomains on which Theorem 2 continues to hold. More generally, little is known about the validity of most results in the additive theory of output sharing on restricted domains, including Lemma 1, which is crucial for the proofs of our two theorems.

4) We repeat that the nearly serial methods form a very small class. In the continuous model (where inputs are measured in real numbers), these methods have no natural counterpart: we conjecture that the five axioms in Theorem 2 characterize precisely Friedman and Moulin's (1999) serial method. Friedman's (2004) representation theorem (stating that the methods meeting Additivity and Dummy are convex combinations of path methods) should prove useful in tackling this conjecture.

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