The Canadian Macroeconomy and the Yield Curve: An Equilibrium-Based Approach

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Abstract

The authors develop and estimate an equilibrium-based model of the Canadian term structure of interest rates. The proposed model incorporates a vector-autoregression description of key macroeconomic dynamics and links them to those of the term structure, where identifying restrictions are based on the first-order conditions that describe the representative investor's optimal consumption and portfolio plan. A remarkable result is that the in-sample average pricing errors obtained with the equilibrium-based model are only slightly larger than those obtained with a far more flexible no-arbitrage model. The gains associated with parsimony become obvious out-of-sample, where the equilibrium model delivers much more accurate predictions, especially for yields with longer-term maturities. The preferred equilibrium model has impulse responses that are consistent with long-term inflation expectations being anchored, so a surprise increase in inflation does not necessarily raise expectations of higher future inflation.

JEL classification: E43, E44, E47, E52

Résumé

Les auteurs élaborent et estiment un modèle d'équilibre général de la structure des taux d'intérêt canadiens, dans lequel la dynamique des principales variables macroéconomiques est représentée sous une forme vectorielle autorégressive et reliée à celle de la structure des taux. Les contraintes d'identification du modèle découlent des conditions du premier ordre qui définissent le plan optimal de consommation et de placement de l'investisseur représentatif. Résultat frappant, l'erreur moyenne de prévision des prix obtenue en échantillon est à peine plus élevée dans le modèle d'équilibre que dans un modèle beaucoup plus souple fondé sur l'absence d'arbitrage. Les gains découlant du caractère parcimonieux du modèle sont très nets au delà de la période d'estimation : le modèle d'équilibre produit des prévisions de qualité bien supérieure hors échantillon, surtout dans le cas des taux d'intérêt à long terme. Les profils de réaction que génère le modèle d'équilibre privilégié cadrent avec un ancrage des attentes d'inflation à long terme, en ce sens qu'une hausse imprévue de l'inflation n'accentue pas nécessairement les attentes d'une augmentation de l'inflation dans l'avenir.

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1. Introduction

Models of the term structure of interest rates have been mostly formulated in continuous time and in an arbitrage-free framework. Typically, bond yields are affine functions of a number of state variables that capture the uncertainty present in the economy. In many specifications, the state variables are unobserved. Econometrically, the latent factors are extracted from bond prices or yields by either assuming that a few bonds are priced perfectly by the model or by filtering techniques if all bonds are assumed to be priced with error. When three factors are specified, they are often interpreted as the level, slope, and curvature of the yield curve, following Litterman and Scheinkman (1991). Dai and Singleton (2003) and Piazzesi (2003) provide thorough surveys of this class of models.

Recently, several researchers have added observable macroeconomic variables to the latent factors to try to understand the channels through which the economy influences the term structure, and not simply describe or forecast the movements of the term structure. Ang and Piazzesi (2003) and Ang, Dong, and Piazzesi (2004) introduce measures of inflation and real activity as macroeconomic factors. The joint dynamics of these macro factors and the latent factors are captured by vector-autoregression (VAR) models, where identifying restrictions are based on the absence of arbitrage. Models with more macroeconomic structure have also been proposed recently by Hordhal, Tristani, and Vestin (2003), Rudebusch and Wu (2004), and Bekaert, Cho, and Moreno (2003). These models combine the affine arbitrage-free dynamics for yields with a New Keynesian macroeconomic model, which typically consists of a monetary policy reaction function, an output equation, and an inflation equation.

In each of the aforementioned models, risk premiums for the various sources of uncertainty are obtained by specifying time-varying prices of risk that transform the risk-factor volatilities into premiums. The prices of risk, however, are estimated directly from the data without accounting for the fact that investors' preferences and technology may impose some constraints between these prices. Indeed, according to Diebold, Piazzesi, and Rudebusch (2005), "the goal of an estimated no-arbitrage macro-finance model specified in terms of underlying preference and technology parameters (such that the asset pricing kernel is consistent with the macrodynamics) remains a major challenge."

In this paper, we propose an equilibrium-based model that goes some distance towards this goal. We price bonds in an economy where investors derive utility from consumption and an external reference level of consumption. The new feature of this approach—introduced by Garcia, Renault, and Semenov (2002)—is that the reference level is formed by expectations about aggregate per capita consumption and not by looking at past consumption, as in habit models. Therefore, the growth rate of this reference level of consumption, which is what matters for pricing purposes, is made a function of contemporaneous and past variables that are deemed relevant. In our model, the short-term interest rate is considered a main explanatory variable, since we consider it a policy instrument under the influence of the central bank. We also relate the reference level of consumption to inflation and past consumption growth, in order to both capture persistence and measure real activity, and finally to the return on a stock index, in order to link the equity and the bond markets. This forecasting equation for consumption growth, to which a structural preference role is given, is added to other equations for the explanatory variables to form a VAR. The same preference parameters that affect the reference-level growth rate in the stochastic discount factor (SDF) impose restrictions on the pricing kernel, and therefore on the term premiums of bonds at various maturities. In Piazzesi (2003), affine general-equilibrium models are specified with preference shocks that are related to state variables, as in Campbell (1986) and Bekaert and Grenadier (2003). Wachter (2005) also proposes a consumption-based model of the term structure of interest rates, where nominal bonds depend on past consumption growth through habit, and on expected inflation. This model is essentially the same as the habit model of Campbell and Cochrane (1999), but the sensitivity function of the surplus consumption to innovations in consumption is chosen so as to make the risk-free rate a linear function of the deviations of the surplus consumption from its mean. Moreover, Wachter calibrates her model so as to make the nominal risk-free rate in the model equal to the yield on a three-month bond at the mean value of surplus consumption. This model has some similarities with ours, but our modelling for the reference level of consumption is more general and we estimate the model as in the no-arbitrage literature, allowing for a direct comparison.

The dynamic interaction between the macroeconomy and the term structure is explored by Diebold, Rudebusch, and Aruoba (2005) in a Nelson-Siegel empirical model of the term structure, complemented by a VAR model for real activity, inflation, and a monetary policy instrument. They find that the causality from the macroeconomy to yields is much stronger than in the reverse direction. Our model will allow for such an effect of macroeconomic variables on yields, but not the reverse. Allowing for the reverse effect complicates the estimation considerably. Ang, Dong, and Piazzesi (2004) use Markov chain Monte Carlo methods to allow for bidirectional linkages between the macroeconomy and the yields, while imposing no-arbitrage restrictions.

We start by estimating a first-order VAR comprising the short-term rate of interest, the return on the Toronto Stock Exchange (TSX) composite index, the rate of inflation, and the rate of consumption growth. We use a sample of quarterly data covering the period 1962Q1 to 2004Q1. Given the parameter estimates of the VAR model and the Euler conditions for the prices of bonds at some chosen maturities, we can estimate the preference parameters by minimizing the least-square distance between the observed yields and the theoretical yields. In order to incorporate information from the various ends of the yield curve, we choose three maturities—2, 8, and 20 quarters—and use the available yields data from the first quarter of 1986 to the last quarter of 2002. Note that the available yields data begin much later (1986) than the macroeconomic data (1962).

For comparison purposes, we also estimate a no-arbitrage model similar to that used by Ang and Piazzesi (2003) and Ang, Piazzesi, and Wei (2004). In the latter, the authors use an approach that is similar to the method just described, but in a no-arbitrage framework.[2] They first estimate the VAR and use the estimated parameters together with the no-arbitrage bond-yield formulas to estimate the prices of risk that minimize a distance between the theoretical yields and the observed yields. By using the same VAR for the macroeconomic variables, we will be able to assess the relative contributions of the different modelling strategies. The equilibrium approach involves far fewer parameters than the no-arbitrage approach. Therefore, one can expect that the no-arbitrage approach will perform better in-sample than the equilibrium approach, and that the reverse will be true out-of-sample, especially for longer maturities. When we assess the in-sample and out-of-sample pricing errors associated with the two methods, our results conform to this expected behaviour. What is more surprising is the overall good performance of the equilibrium model: the absolute pricing errors in-sample are very close to the errors of the no-arbitrage model, despite a large difference in the number of estimated parameters between the two models. It should be stressed that, in the equilibrium model, we maintain that bond prices are determined by the same preferences at short and long horizons, true to the spirit of a structural model. If the goal was simply to reduce pricing errors, we could adjust different preference parameters for each horizon.

The 1986–2002 period over which we estimate the previous term-structure models contains a recession at the beginning of the nineties, during which the behaviour of the term structure of interest rates is markedly different than in the rest of the sample period. As Figure 3 shows, an inversion of the yield curve starts before that recession and ends before

[2] Ang, Piazzesi, and Wei (2004) propose such a sequential estimation strategy. What we gain in flexibility by proceeding in such a sequential manner, we may lose in efficiency of the estimators. Joint estimation is possible, but will add a significant layer of complexity.

the recession finishes. This episode bears a potentially important weight on the average yield curve in our relatively short sample. More generally, one would like to capture in a model this change of behaviour in the yield curve at the approach of recessions. Recently, several studies have built models of the term structure with regime changes. Bansal and Zhou (2002), Dai, Singleton, and Yang (2003), and Ang and Bekaert (2003) have proposed several models with regime shifts where the transition between regimes is driven by an unobservable state variable following a Markov process. One of the main difficulties in these models is associated with the joint inference about the state of the economy and the parameters of the term-structure model. Bansal and Zhou (2002) adopt the efficient method of moments with an approximate linearized version of their model, whereas Dai, Singleton, and Yang (2003) and Ang and Bekaert (2003) estimate their models by maximum likelihood. We propose a very flexible and easy-to-estimate tool to forecast the yield curve, allowing for changes in regimes. Our model is built on various building blocks: a forecasting model for the probability of a recession, a regime-switching VAR model for our macroeconomic variables of interest, and an equilibrium term-structure model with only a limited number of preference parameters to estimate.

We introduce regimes in our model with a consumption reference level by specifying a regime-switching VAR. Contrary to most of the literature on Markov regime switching, however, we assume that both the investor and the modeller know the state of the economy. To this end, we define the current depth of recession, CDR_t, as the gap between the current level of output and the economy's historical maximum level; i.e., $CDR_t = \max\{Y_{t-j}\}_{j\geq 0} - Y_t$, following Beaudry and Koop (1993). The economy is said to be in recession when $CDR_t > 0$, and in expansion when $CDR_t = 0$. This way of measuring the business cycle, instead of treating it as unobservable and filtering it through an algorithm, has recently been a topic of debate between Harding and Pagan (2002, 2003) on the one hand, and Hamilton (2003) on the other. Harding and Pagan argue that an algorithm based on dating rules provides a good approximation of the business cycle chronology determined by the U.S. National Bureau of Economic Research (NBER) and find little value added to Markov switching models to determine cyclical turning points. Hamilton, however, asserts that the Harding-Pagan criterion is simply a rule that one applies, irrespective of the data or one's purpose, while the statistical model underlying the Markov switching dates holds that there is a real event (an economic recession) that either occurred or did not.

The use of a dating rule in the context of our term-structure model has two main advantages. First, by identifying each period as either a recession or a boom, it allows us to estimate a probit model with economic variables that has some forecasting power, and to use it to forecast the yield curve. Second, this estimated probability function makes the regime-switching probabilities state and time dependent. Most discrete-time term-structure models with regimes suppose that this probability is constant. Dai, Singleton, and Yang (2003) are, to our knowledge, the only exception. They show that state dependence of the transition probabilities matters most in the persistence of regimes. While both regimes are highly persistent in the empirical literature on regime-switching models of interest rates with constant probability (Ang and Bekaert 2003 and Bansal and Zhou 2002), Dai, Singleton, and Yang (2003) find that high-volatility regimes are less persistent than low-volatility regimes, as in descriptive models of bond yields.

When we estimate this regime-switching model, the pricing errors in-sample appear to be much larger than the two models without any change in regime. This poor performance is probably due to the fact that we have only one significant recession episode in our yields sample. A true test of the model would necessitate a longer data set dating back to the beginning of the seventies. Nevertheless, the modelling strategy we propose could be a useful tool to predict recessions and incorporate a switching VAR in a term-structure model.

To conclude the empirical assessment of the equilibrium model, we use the VAR specification for the dynamics of the macroeconomic variables to compute impulse-response functions for the yields and a long-short spread. We can then study the impact of an innovation in, say, the inflation rate on the yield structure. A striking feature is the highly persistent effects that shocks have on the yield curve. In each case, the effects are seen to persist for more than 20 quarters before showing signs of significant mean reversion. As one might expect, shocks to the short rate and the inflation rate produce the highest responses. The impulse responses are consistent with a monetary reaction that raises the short end of the yield curve in response to positive shocks to output and inflation. Our results reveal that the short end of the yield curve is more sensitive than the long end to such reactions.

The rest of this paper is organized as follows. Section 2 describes the equilibrium model with a reference level of consumption that will be used to price bonds. We also specify the dynamics of the macroeconomic variables that will influence the yields and allow for regime changes. In addition, we propose a model to forecast the probabilities of recessions. Section 3 is dedicated to model estimation and evaluation. We specify the data sources and the econometric method used to estimate the preference parameters and ultimately to compute the yields. We report the pricing errors for the various specifications as well as the impulse-response functions. Section 4 offers some conclusions. An appendix provides the VAR estimation results.

2. An Equilibrium Model with a Reference Level

Most models of the term structure are specified in a no-arbitrage setting, where the link between the objective data-generating measure and the risk-neutral measure is specified exogenously and is not tied to preference parameters. Piazzesi (2003) describes affine general-equilibrium models within the context of a representative-agent endowment economy. Models in this category[3] are represented by a utility function where the agent consumes an endowment process and receives exogenous preference shocks. These shocks are tied to a vector of state variables. As we will show below, this representation is in the spirit of our model with a reference level of consumption.

2.1 Equilibrium bond prices

We adopt a consumption-based asset pricing proposed by Garcia, Renault, and Semenov (2002), whereby the investor derives utility from consumption relative to some reference consumption level as well as from this level itself. The one-period real SDF defined by this model is

$$m_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{\gamma-\varphi}, \tag{1}$$

where γ is the curvature parameter for consumption C_t relative to S_t, a time-varying reference consumption level. The parameter φ controls the curvature of utility over this benchmark level. The future evolution of the reference level is constrained to coincide with real aggregate per capita consumption in terms of conditional expectations:

$$E_t[S_{t+h}] = E_t[\bar{C}_{t+h}],\ \ for\ all\ h \geq 0. \tag{2}$$

The investor can include in their assessment macroeconomic variables that belong to their information set at time $t+h$.

In nominal terms, the one-period SDF is given by

$$m^{\$}_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{\gamma-\varphi} \pi^{-1}_{t+1}, \tag{3}$$

[3]Piazzesi (2003) refers in particular to Campbell (1986), Bekaert and Grenadier (2003), and Wachter (2005). In each of these papers, a stochastic process is specified for consumption or surplus consumption as defined in Campbell and Cochrane (1999).

where π_{t+1} is the gross rate of inflation between periods t and $t+1$.

An important part of the modelling strategy is to determine the variables that the investor considers for characterizing the growth rate of the reference level of consumption:

$$s_{t+1} = a + x_{t+1} b, \tag{4}$$

where $s_{t+1} = \log S_{t+1}/S_t$ and x_{t+1} represents a vector of variables, current and past, deemed to forecast the growth of the reference level. To obtain an estimate of this growth rate, estimates of a and b are needed along with the values of the variables x_{t+1}. Using the equality of expectations in (2) at horizon 1, we can write a regression equation in terms of consumption growth:

$$c_{t+1} = a + x_{t+1} b + \varepsilon_{t+1}, \tag{5}$$

where $c_{t+1} = \log C_{t+1}/C_t$. Note that the bar over real aggregate consumption is left out, since it is the same as the consumption of the representative investor. Garcia, Renault, and Semenov (2002) show that, by using different specifications for the reference level, they can recover the SDFs associated with the three main strands of consumption-based asset-pricing models: habit formation (Constantinides 1990; Campbell and Cochrane 1999), recursive utility (Epstein and Zin 1989), and loss aversion (Barberis, Huang, and Santos 2001). Models of consumption with a reference level may be rationalized by a behavioural model, as in Kószegi and Rabin (2004).

For the purpose of pricing bonds, we assume that the investor's forecasts of consumption growth are based on

$$c_{t+1} = b_0 + b_1 R_t^s + b_2 R_t^m + b_3 \pi_t + b_4 c_t + \varepsilon_{t+1}, \tag{6}$$

where $R_t^m = \log P_t^m / P_{t-1}^m$ represents the (log) return on the market portfolio, $\pi_t = \log P_t^c / P_{t-1}^c$ is the rate of inflation, and R^s is the short-term interest rate. For the purpose of modelling the term structure of interest rates, the inclusion of the short rate is obviously essential. It captures the fact that the short rate is a policy instrument under the influence of the central bank. Instead of using directly the Bank Rate, which has the behaviour of a step function, we prefer to include the yield to maturity of a one-period bond; that is, $R_t^s = -\log P_t^s$, where P^s is the bond price. Inclusion of inflation is essential to price nominal bonds and longer-term maturities. The other variables (c_t, R_t^m) are less essential, but they provide a way to include a more traditional, strictly consumption-based reference level (as in external habit models) and a link to equity markets. Although we do not price equities in the current paper, such an equation for the growth of the reference level could be used to price equities (see Garcia, Renault, and Semenov 2002).

The joint dynamics governing the evolution of the explanatory variables that appear in (5) are modelled as a first-order vector autoregression (VAR) in $Y_t = (R_t^s, R_t^m, \pi_t, c_t)'$. Ang, Piazzesi, and Wei (2004) also specify a VAR in terms of observables, and include the short rate, the term spread, the inflation rate, and output growth. The uniqueness of the Cholesky decomposition used to compute impulse-response functions is with respect to the model-consistent ordering of the variables within the vector Y_t. The first-order VAR that governs the evolution of these variables is written as

$$Y_t = \mu + \Phi Y_{t-1} + \varepsilon_t, \tag{7}$$

where $E[\varepsilon_t \varepsilon_t'] = \Omega = \Sigma\Sigma'$. With the specification of the reference level in (5), the SDF can be rewritten as

$$m_{t+1}^{\$} = \delta \exp(b_0\kappa) \left(P_t^s\right)^{-b_1\kappa} \left(\frac{P_t^m}{P_{t-1}^m}\right)^{b_2\kappa} \left(\frac{P_t^c}{P_{t-1}^c}\right)^{b_3\kappa} \left(\frac{P_{t+1}^c}{P_t^c}\right)^{-1} \left(\frac{C_t}{C_{t-1}}\right)^{b_4\kappa} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \tag{8}$$

where $\kappa = \gamma - \varphi$. Note that the SDF in (8) is well defined, in that it can only be positive. The price of a nominal bond that pays one dollar at time $t+n$ is given by

$$B(t, t+n) =$$

$$E_t\left[\delta^* \left(P_{t,t+n-1}^s\right)^{-b_1\kappa} \left(\frac{P_{t+n-1}^m}{P_{t-1}^m}\right)^{b_2\kappa} \left(\frac{P_{t+n-1}^c}{P_{t-1}^c}\right)^{b_3\kappa} \left(\frac{P_{t+n}^c}{P_t^c}\right)^{-1} \left(\frac{C_{t+n-1}}{C_{t-1}}\right)^{b_4\kappa} \left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\right], \tag{9}$$

where $\delta^* = \delta \exp(nb_0\kappa)$ and $P_{t,t+n-1}^s = \prod_{i=0}^{n-1} P_{t+i}^s$. Given the VAR specification in (7), all the variables entering the bond-pricing equation in (9) can be forecasted conditional on time-t information. Therefore, bond prices are given by

$$B(t, t+n) = \delta^* \exp\left(\bar{M}\mu + M_t Y_t + \frac{1}{2}\sigma_1^2\right), \tag{10}$$

where $\bar{M} = M_1 + M_2$ and $\sigma_i^2 = V_{i,1} + V_{i,2}$ with

$$
\begin{aligned}
M_1 &= (J_2 + J_1) \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \Phi^{n-j-1}, \\
M_2 &= J_1 \sum_{i=1}^{n} \Phi^{n-i}, \\
M_t &= (J_2 + J_1) \sum_{i=1}^{n-1} \Phi^{i} + J_1 \Phi^{n} + J_2, \\
V_1 &= M_1 \Omega M_1', \\
V_2 &= M_2 \Omega M_2',
\end{aligned}
$$

and where $J_1 = [0, 0, -1, -\gamma]$ and $J_2 = \kappa[b_1, b_2, b_3, b_4]$.

The modelling of the reference level may appear arbitrary, since one chooses the variables that may determine the benchmark level of consumption. Whichever variables are chosen, however, they enter the SDF in a restricted way, since they are affected by a common preference expression ($\kappa = \gamma - \varphi$), which imposes testable restrictions on asset prices. One can use the usual Euler conditions on several asset returns together with equation (5) to infer the preference parameters. One can also test the model with a J-test or an asymptotically more appropriate test that accounts for weak instruments (see Stock and Wright 2000 and Yogo 2004). Garcia, Renault, and Semenov (2002) estimate and test several such consumption-based capital-asset-pricing models with a reference level with returns on Treasury bills and on equities.

2.2 No-arbitrage bond prices

The described equilibrium model links the dynamics of the term structure of interest rates to macroeconomic variables. Ang and Piazzesi (2003) also establish such a link through a no-arbitrage model of the term structure. The equilibrium approach taken here and the no-arbitrage approach both incorporate macroeconomic dynamics by specifying a VAR. The fundamental difference lies in the way identification is achieved. In the approach advocated here, identifying restrictions are based on the first-order conditions that describe the representative investor's optimal consumption and portfolio plan. In Ang and Piazzesi's approach, identifying restrictions are based only on the absence of arbitrage.

Ang and Piazzesi (2003) and Ang, Piazzesi, and Wei (2004) assume that the SDF follows a conditionally log-normal process:

$$m^{\$}_{t+1} = \exp\left(-R^s_t - \frac{1}{2}\lambda'_t\lambda_t - \lambda'_t\varepsilon_{t+1}\right),$$

where λ_t are the time-varying market prices of risk associated with the sources of uncertainty, ε_t. The vector λ_t is a linear function of Y_t:

$$\lambda_t = \lambda_0 + \lambda_1 Y_t,$$

where Y_t is described by (7) so that λ_0 is a 4×1 vector and λ_1 is a 4×4 matrix.

Bond prices are given by

$$B^{na}(t, t+n) = \exp(A_n + B_n Y_t), \tag{11}$$

where the coefficients A_n and B_n are defined recursively by

$$\begin{aligned} A_n &= A_{n-1} + B'_{n-1}(\mu - \Sigma\lambda_0) + \frac{1}{2}B'_n\Sigma\Sigma' B_n, \\ B_n &= (\Phi - \Sigma\lambda_1)B'_{n-1} - I'_1, \end{aligned}$$

with $I_1 = [1, 0, 0, 0]$. The initial conditions are $A_1 = 0$ and $B_1 = -I'_1$. The above definitions of A_n and B_n incorporate the no-arbitrage restrictions; see Ang and Piazzesi (2003) and Ang, Piazzesi, and Wei (2004) for details.

Note that the identifying restrictions in the equilibrium approach appear directly in the definition of the SDF, so that *any* financial asset can be consistently priced and linked to macroeconomic dynamics. On the other hand, Ang and Piazzesi's no-arbitrage approach applies only to bonds.

2.3 Adding regimes to the equilibrium model

The equilibrium specification is easily generalized to accommodate potentially important regime shifts by considering a VAR of the form:

$$Y_t = \mu_t + \Phi Y_{t-1} + \varepsilon_t, \tag{12}$$

where $\mu_t = \mu_0(1 - U_t) + \mu_1 U_t$; the variable U_t is an observed binary state variable equal to one when the economy is in recession, and zero otherwise. Conditional on the realizations of the state variable, $U^t_1 = \{U_\tau\}^t_{\tau=1}$, the error terms are independently distributed as

$$\varepsilon_t \mid U^t_1 \sim \mathrm{N}(0, \Omega_t), \tag{13}$$

with the time-varying covariance matrix $\Omega_t = \Omega_0(1 - U_t) + \Omega_1 U_t$, where

$$\Omega_i = \begin{bmatrix} \sigma_i^{s\ 2} & \sigma_i^{s,m} & \sigma_i^{s,\pi} & \sigma_i^{s,c} \\ \sigma_i^{m,s} & \sigma_i^{m\ 2} & \sigma_i^{m,\pi} & \sigma_i^{m,c} \\ \sigma_i^{\pi,s} & \sigma_i^{\pi,m} & \sigma_i^{\pi\ 2} & \sigma_i^{\pi,c} \\ \sigma_i^{c,s} & \sigma_i^{c,m} & \sigma_i^{c,\pi} & \sigma_i^{c\ 2} \end{bmatrix}, \tag{14}$$

for $i = 0, 1$; the correlations are also free to vary over time. The state variable is governed by a two-state first-order Markov chain with a time-varying transition probability matrix:

$$\Pi_t = \begin{bmatrix} p_{00,t} & p_{01,t} \\ p_{10,t} & p_{11,t} \end{bmatrix}, \tag{15}$$

where $p_{ij,t} = \Pr[U_{t+1} = j \mid X_t, U_t = i]$ and $\sum_{j=0}^{1} p_{ij,t} = 1$. The transition probabilities may depend on other factors included in the vector X_t.

The price of a nominal bond that pays one dollar at time $t + n$ is still given by the expression in (9), where

$$\delta^* = \delta \exp(n b_0 \kappa), \tag{16}$$

$$\left(P^s_{t,t+n-1}\right)^{-b_1\kappa} = \exp(b_1 \kappa I_1 \sum_{i=0}^{n-1} Y_{t+i}) \tag{17}$$

$$\left(\frac{P^m_{t+n-1}}{P^m_{t-1}}\right)^{b_2\kappa} = \exp(b_2 \kappa I_2 \sum_{i=0}^{n-1} Y_{t+i}) \tag{18}$$

$$\left(\frac{P^c_{t+n-1}}{P^c_{t-1}}\right)^{b_3\kappa} = \exp(b_3 \kappa I_3 \sum_{i=0}^{n-1} Y_{t+i}), \tag{19}$$

$$\left(\frac{P^c_{t+n}}{P^c_{t}}\right)^{-1} = \exp(-I_3 \sum_{i=1}^{n} Y_{t+i}), \tag{20}$$

$$\left(\frac{C_{t+n-1}}{C_{t-1}}\right)^{b_4\kappa} = \exp(b_4 \kappa I_4 \sum_{i=0}^{n-1} Y_{t+i}), \tag{21}$$

$$\left(\frac{C_{t+n}}{C_{t}}\right)^{-\gamma} = \exp(-\gamma I_4 \sum_{i=1}^{n} Y_{t+i}), \tag{22}$$

with $I_1 = [1, 0, 0, 0]$, $I_2 = [0, 1, 0, 0]$, $I_3 = [0, 0, 1, 0]$, and $I_4 = [0, 0, 0, 1]$; the VAR in (12) implies that

$$\sum_{i=1}^{n} Y_{t+i} = \sum_{i=1}^{n-1} Y_{t+i} + Y_{t+n}, \tag{23}$$

$$\sum_{i=0}^{n-1} Y_{t+i} = \sum_{i=1}^{n-1} Y_{t+i} + Y_t, \tag{24}$$

$$\sum_{i=1}^{n-1} Y_{t+i} = \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \Phi^{n-j-1} \left(\mu_{t+i} + \varepsilon_{t+i}\right) + \sum_{i=1}^{n-1} \Phi^i Y_t, \tag{25}$$

$$Y_{t+n} = \sum_{i=1}^{n} \Phi^{n-i} \left(\mu_{t+i} + \varepsilon_{t+i}\right) + \Phi^n Y_t, \tag{26}$$

where Φ^0 is set equal to the 4×4 identity matrix.

Multiplying (16) through (22) together and taking the expectation conditional on time-t information, we obtain

$$B(t, t+n) = E_t \left[\delta^* \exp \left(\bar{M}_1 \mu_0 + \bar{M}_2 \mu_1 + M_t Y_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \right) \right], \tag{27}$$

where $\bar{M}_i = M_{i,1} + M_{i,2}$ and $\sigma_i^2 = V_{i,1} + V_{i,2}$, with

$$\begin{aligned}
M_{1,1} &= (J_2 + J_1) \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \Phi^{n-j-1}(1 - U_{t+i}), \\
M_{1,2} &= J_1 \sum_{i=1}^{n} \Phi^{n-i}(1 - U_{t+i}), \\
M_{2,1} &= (J_2 + J_1) \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \Phi^{n-j-1} U_{t+i}, \\
M_{2,2} &= J_1 \sum_{i=1}^{n} \Phi^{n-i} U_{t+i}, \\
M_t &= (J_2 + J_1) \sum_{i=1}^{n-1} \Phi^i + J_1 \Phi^n + J_2, \\
V_{1,1} &= M_{1,1} \Omega_0 M_{1,1}', \\
V_{1,2} &= M_{1,2} \Omega_0 M_{1,2}', \\
V_{2,1} &= M_{2,1} \Omega_1 M_{2,1}', \\
V_{2,2} &= M_{2,2} \Omega_1 M_{2,2}',
\end{aligned}$$

and where $J_1 = [0, 0, -1, -\gamma]$ and $J_2 = \kappa[b_1, b_2, b_3, b_4]$.

The value of the expectation on the right-hand side of (27) can be written in matrix form as

$$E_t\left[\delta^* \exp\left(\bar{M}_1\mu_0 + \bar{M}_2\mu_1 + M_tY_t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\right)\right] = \delta^* I_t\left(D_0(\prod_{i=1}^{n-1}\Pi_t D_i)\Pi_t D_n\right) l', \quad (28)$$

with $I_t = [1-U_t, U_t]$, $l = [1,1]$, and

$$D_0 = \text{Diag}\left[\exp(M_tY_t),\ \exp(M_tY_t)\right],$$

$$D_i = \text{Diag}\left[\exp\left(M^*\mu_0 + \frac{1}{2}\sigma_1^{*2}\right),\ \exp\left(M^*\mu_1 + \frac{1}{2}\sigma_2^{*2}\right)\right],$$

$$D_n = \text{Diag}\left[\exp\left(J_1\mu_0 + \frac{1}{2}J_1\Omega_0J_1'\right),\ \exp\left(J_1\mu_1 + \frac{1}{2}J_1\Omega_1J_1'\right)\right],$$

where

$$\begin{aligned} M^* &= (J_2+J_1)\sum_{j=i}^{n-1}\Phi^{n-j-1} + J_1\Phi^{n-i}, \\ \sigma_1^{*2} &= M^*\Omega_0M^{*\prime}, \\ \sigma_2^{*2} &= M^*\Omega_1M^{*\prime}. \end{aligned}$$

The model-implied n-period yield, $\hat{y}^n(U_t) = -\log B(t,t+n)/n$, is thus given by

$$\hat{y}^n(U_t) = \begin{cases} -\frac{1}{n}\log\left[\delta^* I_t\left(D_0(\prod_{i=1}^{n-1}\Pi_tD_i)\Pi_tD_n\right)l'\right] & \text{if } B(t,t+n) > 1, \\ 0 & \text{if } 0 \le B(t,t+n) \le 1, \end{cases} \quad (29)$$

where the dependence on the state operative at the time of pricing is made explicit. The developed partial-equilibrium model does not explicitly incorporate money, so a priori it does not exclude negative interest rates. This can be explained by the fact that if the cost of storing currency exceeds that of storing other financial assets, then nominal interest rates could become negative. On the other hand, if the cost of storing cash is zero and non-monetary assets are viewed as perfect substitutes, then nominal interest rates cannot become negative. We impose the lower zero bound on bond yields in (29) since, in reality, negative nominal interest rates occur very rarely.[4]

Two arbitrage-based models are close in spirit to our model. Bansal and Zhou (2002) propose a model of the term structure of interest rates with regime switches. Bekaert and

[4]One recent occurrence is the slightly negative interest rates on short-term Japanese government bonds in late 1998.

Grenadier (2003) propose a bond- and stock-pricing model in an affine economy. We will sketch here the main difference between these two models and our approach. Ignoring the regime-switching feature, Bansal and Zhou (2002) start with the Lucas (1978) model, a particular case of our model with $\gamma = \varphi$, where $M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$. The next step is to relate the SDF to the return of an asset that delivers the consumption stream: $R_{c,t+1} = (M_{t+1})^{-1}$. Bansal and Zhou (2002) then assume a stochastic process for the logarithm of this return: $r_{c,t+1} = x_t + \frac{\lambda}{\sigma}\frac{x_t}{2} + \sqrt{x_t}\frac{\lambda}{\sigma}u_{t+1}$, where x_t is a latent variable following a Cox, Ingersoll, and Ross (1985) process. By absence of arbitrage and the normality of u_{t+1}, the return on a one-period safe asset enters the SDF as

$$M_{t+1} = \exp(-r_{ft} - (\frac{\lambda}{\sigma})^2 \frac{x_t}{2} - \sqrt{x_t}\frac{\lambda}{\sigma}u_{t+1}). \qquad (30)$$

The SDF for the economy with regime shifts is similar, except that the λ and σ are made state dependent. In a multivariate setting, Bekaert and Grenadier (2003) exploit a similar arbitrage-based model, but without regime switches, to price both bonds and stocks:

$$\begin{aligned} Y_{t+1} &= \mu + AY_t + \Sigma_{Y_t}\varepsilon_{t+1} \\ m_{t+1} &= \mu_m + \Gamma_m^{'} Y_t + \Sigma_{m_t}\varepsilon_{t+1}, \end{aligned} \qquad (31)$$

where Y_{t+1} is a vector of observable and latent state variables, the matrices Σ_{Y_t} and Σ_{m_t} are such that the processes for Y_t and m_t are a combination of Vasicek and square-root processes, and ε_{t+1} is standard normal. In their applications of this general specification, Bekaert and Grenadier (2003) include a dividend-growth process, a latent variable, and inflation in their state variables, and choose several specifications for m_t, including Campbell and Cochrane's (1999) habit-formation model.[5] In both these models, the latent process is used to capture the features of the short-rate process. Indeed, for estimation, Bekaert and Grenadier (2003) use the nominal interest rate to filter the latent process. In our model, the short rate enters directly into the SDF through the preferences. In Wachter (2005), who also uses the habit-based model of Campbell and Cochrane (1999), the equilibrium short-term interest rate is restricted to be a linear function of surplus consumption. It is further calibrated to be equal to the nominal interest rate in steady state, when surplus consumption equals its long-term mean.

Our model has a number of advantages for capturing the yield curve. First, by making the short rate exogenous, the model captures the fact that central banks of industrialized

[5] Although it is not our purpose here, Campbell and Cochrane's (1999) model could be accommodated in our general specification, as shown by Garcia, Renault, and Semenov (2002).

countries may affect the short end of the yield curve. The model provides a way to introduce monetary policy in the form of reaction functions, which is consistent with equilibrium. Second, our model links bond prices with the real and monetary sides of the economy while accounting for business fluctuations. Recently, several papers (Bekaert, Cho, and Moreno 2003; Hordahl, Tristani, and Vestin 2003; Rudebusch and Wu 2003) have appended a term-structure model to a New Keynesian macro model. Our proposed equilibrium approach to capture the dynamic interactions between the macroeconomy and the term structure is more parsimonious and is built only on observable variables. Moreover, the risk premiums are fully pinned down by the risk processes and the preference parameters, while they are left free in the arbitrage-based models of the term structure used in these papers. A third advantage of our model is that it provides a link between the stock market and the bond market by including a return on a stock market index in the equation that determines the growth of benchmark consumption. The regime-switching nature of the model makes it possible for the correlation between bonds and stocks to change with the business cycle. Fourth, as in any equilibrium model, we can price any asset and, in particular, derivatives such as swaps, futures, and options on interest rates.[6]

2.4 Forecasting the probabilities of recession

An important development in business cycle modelling is the class of Markov-switching models introduced by Hamilton (1989), where regimes are treated as unobserved. Another approach is to represent transitions between recessions and expansions by an observed binary-choice model; see Estrella and Hardouvelis (1991), Estrella and Mishkin (1998), Birchenhall et al. (1999), and Pederzoli and Torricelli (2005) for applications of this approach. In particular, Birchenhall et al. (1999) examine the performance of logistic procedures in reproducing the NBER business cycle chronology. Their model provides substantially more accurate predictions about business cycle regimes than Markov switching specifications. This evidence motivates the choice made here.

In Canada, there is no widely recognized counterpart to the NBER committee responsible for the determination of cyclical turning points. A Canadian business cycle chronology is obtained, following Beaudry and Koop (1993), by defining the current depth of recession, CDR_t, as the gap between the current level of output and the economy's historical maximum level; i.e., $CDR_t = \max\{Y_{t-j}\}_{j\geq 0} - Y_t$. The economy is said to be in recession

[6]Garcia, Luger, and Renault (2003) estimate an equilibrium model with regime-switching and Epstein and Zin (1989) preferences using prices on Standard and Poor's 500 options.

when $CDR_t > 0$ and in expansion when $CDR_t = 0$. The current length of recession, CLR_t, is said to be equal to h when $CDR_{t-j} > 0$ for $j = 0, 1, ..., h$. The level of output and the CDR_t variable in the case of Canada over the period 1962Q1–2004Q1 are plotted in Figure 1, where the output measure is quarterly seasonally adjusted GDP in chained 1997 dollars.

The above definitions of recessions and expansions provide a simple and well-defined alternative to subjective dating practices, and in the case of the United States they yield a business cycle chronology that corresponds closely to that of the NBER (Hess and Iwata 1997). The dating algorithm used here to determine business cycles is similar to the two-sided algorithm advocated by Harding and Pagan (2002), except that ours is one-sided, which makes it particularly useful for out-of-sample forecasting purposes. These non-parametric algorithms also avoid many of the difficulties in using Markov switching models to determine turning points; see the debate between Harding and Pagan (2002, 2003) and Hamilton (2003).

The probit model that we use assumes an underlying latent variable, U_t^*, for which there exist realizations of an indicator variable, U_t, denoting the occurrence of a recession. Let $U_t = 1$ if $CDR_t > 0$ and $U_t = 0$ if $CDR_t = 0$, so that U_t^* represents the state of the economy.

The unobserved variable, U_t^*, is related to lagged values of the current depth and current length of recession, such that

$$U_t^* = X_t\beta + \gamma_1 CDR_{t-1} + \gamma_2 CLR_{t-1} + \varepsilon_t,$$

where the row vector X_t contains lagged values of other macroeconomic and financial variables believed to have predictive ability for future recessions; e.g., lags of interest rates and interest rate spreads. The inclusion of CDR_{t-1} and CLR_{t-1} in the specification captures the notion of recession intensity and duration dependence. The error terms, ε_t, are assumed to be independently distributed according to a standard normal distribution. Note that the assumption of unit variance is innocuous. This follows from the model-implied observations:

$$\begin{aligned} U_t &= 1 \quad \text{if } U_t^* > 0, \\ U_t &= 0 \quad \text{if } U_t^* \leq 0, \end{aligned}$$

where the value of the indicator variable, U_t, is seen to depend only on the sign of U_t^*, and not on its scale. The model specification, along with the symmetry of the normal distribution, implies that

$$\Pr[U_t = 1 \mid X_t, U_{t-1}] = \Phi[X_t\beta + \gamma_1 CDR_{t-1} + \gamma_2 CLR_{t-1}],$$

where $\Phi[\cdot]$ is the cumulative distribution function of the standard normal distribution.

Let $W_t\theta = X_t\beta + \gamma_1 CDR_{t-1} + \gamma_2 CLR_{t-1}$. The log-likelihood function of the observed data is

$$\log L(\theta) = \sum_{t=1}^{T} U_t \log \Phi(W_t\theta) + \sum_{t=1}^{T} (1 - U_t) \log \left(1 - \Phi(W_t\theta)\right), \tag{32}$$

and the corresponding information matrix is

$$I(\theta) = \sum_{t=1}^{T} \frac{\phi(W_t\theta)^2}{\Phi(W_t\theta)\left(1 - \Phi(W_t\theta)\right)} W_t' W_t,$$

where $\phi(\cdot)$ is the density function of the standard normal distribution. Maximum-likelihood estimates, $\hat{\theta}$, can be obtained by maximizing (32), and the asymptotic covariance matrix can then be estimated by $I(\hat{\theta})^{-1}$. Note that the Hessian, $\partial^2 \log L(\theta)/\partial\theta'\partial\theta$, is negative definite for all values of θ (Maddala 1983, 27). Hence, the method of scoring (Newton's method) will converge to the global maximum of the likelihood function no matter what the starting values are, unless the data are especially badly conditioned.

3. Model Estimation and Evaluation

3.1 Data description

The macroeconomic data used to estimate the VAR are quarterly, covering the period 1962Q1 to 2004Q1. Consumption growth is based on seasonally adjusted total personal expenditures in chained 1997 dollars, and inflation is defined by the corresponding implicit chain prices. The returns on the market portfolio are proxied using the month-to-month changes in the (log) value of the TSX composite index; the log returns are then averaged to obtain quarterly returns. The short rate is also obtained by averaging the monthly 3-month Treasury bill rate.

The data used to estimate the probit model include the 3-month Treasury bill (short) rate and a spread: the difference between the 10-year government bond rate and the 3-month Treasury bill rate. The short rate is monthly, covering the period from January 1962 to March 2004, and the monthly 10-year rate begins only in June 1982. The spread, consequently, is defined only from June 1982 to March 2004. The quarterly rates used are averages of the monthly rates.

The bond data consist of a set of daily zero-coupon yields on Canadian government bonds, obtained from the Bank of Canada. This data set was derived by Bolder and

Gusba (2002) using the Merrill Lynch exponential spline model.[7] In the described yield-curve model, one period corresponds to one quarter. The bond data, therefore, were aggregated up to the quarterly frequency by averaging the daily yields. The result is a set of 68 quarterly observations, with considered maturities of 2, 4, 8, 12, 16, and 20 quarters, from 1986Q1 to 2002Q4. Table 1 provides summary statistics of the yield data at the quarterly frequency.

3.2 Estimation results

Estimation results for the probit model are reported in Table 2 for three specifications, including lags of the short rate (Short) and the yield spread (Spread). The results of the likelihood-ratio test show that Specification 2—with only spread lags—is so poorly conditioned that it achieves a log-likelihood value that is less than that of the restricted form containing only a constant term. Specification 1 includes only lags of the short rate, while Specification 3 includes spread lags in addition to those of the short rate. Both of these easily pass the likelihood-ratio test, although Specification 1 appears statistically more significant, judging by the drastically smaller p-value (1.59×10^{-5} for Specification 1 versus 0.037 for Specification 3). This result is consistent with those of Ang, Piazzesi, and Wei (2004), who also find that the short rate has more predictive power for output growth than any term spread in the case of the United States. The intensity dependence of Canadian recessions is shown by the high individual significance of CDR_{t-1}.

An analog to the R^2 in a conventional regression model is the likelihood-ratio index, $LRI = 1 - \log L / \log L_0$, where $\log L_0$ is the log-likelihood computed with only a constant term. The LRI for Specification 1 is 0.19, while that for Specification 3 is 0.11. Another useful summary of the predictive ability of the probit model is a 2×2 table of the hits and misses of a prediction rule of the form:

$$\begin{aligned} \hat{U} &= 1 \quad \text{if } \Phi(W_t\hat{\theta}) > 0.5, \\ \hat{U} &= 0 \quad \text{if } \Phi(W_t\hat{\theta}) \leq 0.5. \end{aligned}$$

Tables 3 and 4 show the number of hits and misses for Specifications 1 and 3, respectively. The overall evidence in Tables 2–4 and the principle of parsimony suggests that Specification 1 should be preferred over Specification 3. The predicted probabilities of recession from the preferred specification are shown in Figure 2.

[7]Bolder, Johnson, and Metzler (2004) describe a similar database, which will be kept current and be publicly available on the Bank of Canada's website.

The estimated probit model paves the way for the estimation of the remaining model parameters; i.e., the parameters of the VAR process that govern the macroeconomic fundamentals in (12), and the preference parameters of the bond-pricing equation that appear in (29). Conditional on the business cycle chronology, $\{U_t\}_{t=1}^{T}$, the parameters of the macroeconomic fundamentals VAR model are easily estimated by dummy variable regressions; the VAR estimates are provided in the appendix. The preference parameters that enter the bond-pricing equation are then estimated, conditional on the VAR estimates, by solving the non-linear least-squares problem:

$$\min_{\{\delta,\gamma,\varphi,b\}} \sum_{t=1}^{T} \sum_{n=1}^{N} \left[\hat{y}^n(U_t) - y_t^n\right]^2 , \tag{33}$$

where y_t^n is the market yield of an n-period bond at time t and $\hat{y}^n(U_t)$ is the corresponding model-implied yield; the choice variables are the three preference parameters—δ, γ, and ϕ—and the parameters of the consumption reference level, $b = [b_0, b_1, b_2, b_3, b_4]$. The model-implied yields are computed according to (29) conditional on the business cycle chronology, the estimated parameters of the probit model, and those of the macroeconomic fundamentals VAR model. In particular, given the value of U_t, the matrix Π_t is completed using

$$\begin{aligned} \hat{p}_{01,t} &= \Phi[X_t\hat{\beta}], \\ \hat{p}_{11,t} &= \Phi[X_t\hat{\beta} + \hat{\gamma}_1 CDR_{t-1} + \hat{\gamma}_2 CLR_{t-1}], \end{aligned}$$

where $\hat{\beta}$, $\hat{\gamma}_1$, and $\hat{\gamma}_2$ are the estimated parameters for the preferred probit specification. The version without regimes is obtained by imposing the constraints $\mu_0 = \mu_1$ and $\Omega_0 = \Omega_1$.

The no-arbitrage model is also estimated by solving a non-linear least-squares problem similar to the one in (33) without regimes; i.e., conditional on the VAR estimates obtained under the constraints $\mu_0 = \mu_1$ and $\Omega_0 = \Omega_1$, the sum of squared differences between actual yields and those implied by the no-arbitrage model, $-\log B^{na}(t, t+n)/n$, is minimized with respect to the parameters in λ_0 and λ_1.

Of course, this step-by-step estimation methodology does not deliver the most statistically efficient estimates. On the other hand, its computational simplicity is a considerable advantage, especially when the models need to be updated on a regular basis. In order to incorporate information from the various ends of the yield curve, we estimate the models based on the observed behaviour of the 2-, 8-, and 20-quarter yields. Figure 3 plots the time series of these three yields.

Table 5 reports the estimation results for the no-arbitrage model. Note that all the components of λ_0 appear to be statistically significant, while only the component of λ_1 associated with the short-term interest rate appears to be significant.

Table 6 reports the estimation results for the equilibrium model with and without regimes. The model without regimes is estimated for both the standard expected utility case ($\gamma = \varphi$) and the case where utility depends on a reference level of consumption. In these two cases, the preference parameters γ (and ϕ in the reference level case) are significantly different from 1. In the reference level model with regimes, γ and ϕ are not individually significant, but their difference, κ, which incorporates the covariance between them, is highly significant. In the version without regimes, all the coefficients in the consumption reference level, except for the one associated with lag consumption, are significant, whereas only the intercept and the coefficient of the short rate remain significant when regimes are introduced.

Table 7 reports summary statistics of the absolute pricing errors (in percentage points) for the various specifications considered. It is immediately clear that relaxing the standard expected utility constraint vastly improves the fit of the equilibrium model. This is not surprising, given that the short rate plays no role in the equilibrium bond-pricing formula under standard expected utility. A striking result that emerges from Table 7 is that allowing for regimes does not lead to any improvements in terms of pricing errors. A possible explanation for this is that the yields data span only one major recession—from 1990Q2 to 1993Q2 (Figure 1). The equilibrium model without regimes fares well against the no-arbitrage model, especially at longer maturities. This result is even more impressive when one considers that the equilibrium model has only 8 parameters compared with the 20 parameters needed for the no-arbitrage model with time-varying prices of risk. Given the similarity of yields at the short end of the yield curve, it is not surprising that the no-arbitrage model overfits for shorter maturities. Figures 4–6 show the resulting fits for maturities of 2, 8, and 20 quarters, respectively.

The equilibrium and no-arbitrage models are further compared in terms of their one-quarter-ahead prediction abilities. For each quarter t, we estimate the VAR model and the two term-structure models using data up to and including quarter t, and then forecast the next quarter's yields using the VAR's forecasts for period $t + 1$. Hence, we use only data available in the information set in period t when forming the forecasts for period $t + 1$. Given that we need at least 20 observations to estimate the no-arbitrage model, prediction abilities are compared over the period 1991Q1–2002Q4, resulting in 48 one-quarter-ahead forecasts.

Table 8 reports summary statistics of the one-quarter-ahead absolute forecast errors (in percentage points). For maturities of 2 and 4 quarters, the no-arbitrage model outperforms the equilibrium model. For all maturities greater than 8 quarters, however, the situation is reversed: the more parsimonious equilibrium model is the better predictor. Figures 7–9 show the time series of predicted 2-, 8-, and 20-quarter yields, respectively.

3.3 Impulse responses of yields

We examine the steady-state dynamics implied by the equilibrium model without regimes by means of impulse-response functions. A standard Cholesky decomposition can be used to identify "structural shocks," which in turn are used to compute impulse-response functions for the VAR in the usual way. Yield-curve impulse-response functions are then computed by feeding the VAR responses into the equilibrium bond-pricing formula.

Figures 10–12 show the impulse responses of yields to one percentage point shocks to the short rate, R^s, the return on the market portfolio, R^m, the rate of inflation, π, and the rate of consumption growth, c. The steady-state 2-quarter yield is 7.823 per cent, while that of the 20-quarter yield is 7.49 per cent. These values imply a slightly inverted steady-state yield curve, as shown by the responses in Figure 12 (the horizontal line represents the steady-state slope). A striking feature of Figures 10–12 is the highly persistent effects the various shocks have. A shock to the short rate has the greatest effect, followed by shocks to inflation, then consumption, and finally the return on the market portfolio.

The impulse responses are consistent with a monetary reaction that raises the short end of the yield curve in response to positive shocks to output and inflation. Figures 10 and 11 show that the short end of the yield curve is more sensitive than the long end to such reactions. Indeed, the more negatively sloped yield curve suggested by Figure 12 is due to the greater increase in shorter-term yields. Such responses are consistent with long-term inflation expectations being anchored, so a surprise increase in inflation, for example, does not necessarily raise expectations of higher future inflation. This result suggests that the central bank enjoys a fairly large degree of credibility and transparency, which concurs with the explicit inflation-control targeting policy of the Bank of Canada.[8]

[8]In the case of the United States—which does not have an explicit inflation target—Diebold, Rudebusch, and Aruoba (2004) find that surprise increases in inflation boost future inflation expectations.

4. Conclusion

Most term-structure models have been formulated in a no-arbitrage setting where bond yields are affine functions of a state vector. In the first generation of models, the short rate was the only state variable in the economy. Recently, term-structure models have modelled the dynamics of bond yields jointly with the dynamics of some key macroeconomic variables. For example, Ang, Piazzesi, and Wei (2005) use GDP growth along with the short rate and a term spread variable to estimate the dynamics of the economy with a quarterly VAR. Bond yield risk premiums are captured by market prices of risk that are linear functions of the state vector. This is tantamount to an exogeneity assumption that ignores the underlying preferences of investors.

We have proposed an equilibrium model that is very close in spirit to this setting, but where risk premiums are determined by the preferences of the representative investor. This specification is based on a model where the investor derives utility with reference to a benchmark consumption level, as in habit-formation models. However, our specification for the dynamic evolution of this reference level is not determined solely by past consumption, as in habit-based equilibrium models. In our model, the growth rate of benchmark consumption is a function of the short rate, inflation, a stock market return, and past consumption growth.

Preferences impose tight restrictions on the SDF and deliver in the end a more parsimonious model than the arbitrage-free model. In our setting, we need to estimate eight parameters for the equilibrium model compared with twenty for the arbitrage-free model. A remarkable result is that the average pricing errors obtained in-sample with the equilibrium model are only slightly larger than the errors obtained with the more flexible arbitrage-free model. The gain associated with parsimony appears out-of-sample, where the equilibrium model provides much smaller errors for yields with maturities longer than one year in a one-quarter-ahead rolling forecast exercise.

The behaviour of the yield curve is distinctly different in recessions, where premiums on long-term bonds tend to be high and yields on short bonds tend to be low. Therefore, we propose a term-structure model that accommodates potentially different dynamics for the macroeconomic variables across the business cycle, together with a forecasting model for recessions. Unfortunately, the value-added of this cyclical model is not apparent in the short sample over which the zero-coupon bond yields are available for Canada. A longer-term database would be required to illustrate the usefulness of distinguishing between booms and recessions. It should be emphasized that the model is built to make estimation and forecasting very easy and robust. This feature makes the model particularly attractive for current analysis and policy simulations.

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Table 1. Summary Statistics of Yield Data

	Maturity in quarters					
	2	4	8	12	16	20
Mean	0.066	0.067	0.069	0.071	0.072	0.074
Std. deviation	0.027	0.025	0.022	0.021	0.019	0.019
Skewness	0.47	0.37	0.28	0.23	0.18	0.15
Kurtosis	2.23	2.12	1.95	1.82	1.74	1.69
Min	0.021	0.025	0.032	0.037	0.041	0.043
Max	0.12	0.13	0.12	0.12	0.11	0.11

Table 2. Estimation Results for the Probit Model

	Specification 1		Specification 2		Specification 3	
Variable	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat
Constant	0.087	0.278	-0.186	-1.512	0.073	0.216
Short_{t-1}	-0.054	-0.366			-0.040	-0.243
Short_{t-2}	0.085	0.363			-2.8×10^{-4}	-0.001
Short_{t-3}	0.015	0.059			-0.104	-0.344
Short_{t-4}	-0.211	-1.296			6.8×10^{-4}	0.004
Spread_{t-1}			0.025	0.278	0.015	0.051
Spread_{t-2}			-0.030	0.278	0.010	0.022
Spread_{t-3}			0.146	0.278	-0.005	-0.011
Spread_{t-4}			0.016	0.278	0.045	0.141
CDR_{t-1}	3.1×10^{-4}	2.929^{*}	7.2×10^{-5}	1.709	3.2×10^{-4}	2.778^{*}
CLR_{t-1}	0.154	1.525	0.101	1.144	0.104	1.120
LR	32.06^{*}		-41.64^{a}		19.21^{*}	

Notes: An asterisk indicates statistical significance at the 5 per cent level. LR is the likelihood-ratio statistic for testing the null hypothesis that all the slope parameters are zero. The 95 per cent critical values of the Chi-square distribution with 6 and 10 degrees of freedom are 12.6 and 18.3, respectively.

[a] Specification 2 is clearly rejected, since it achieves a log-likelihood value less than that of the restricted form that contains only a constant term.

Table 3. Hits and Misses: Specification 1

		Predicted		
		$U = 0$	$U = 1$	Total
Actual	$U = 0$	128	3	131
	$U = 1$	16	18	34
	Total	144	21	165

Table 4. Hits and Misses: Specification 3

		Predicted		
		$U = 0$	$U = 1$	Total
Actual	$U = 0$	124	7	131
	$U = 1$	16	18	34
	Total	140	25	165

Table 5. Parameter Estimates: No-Arbitrage Model

Parameter	Estimate			
λ_0	-0.52*	(0.0030)		
	-0.20*	(0.073)		
	0.86*	(0.045)		
	-0.093*	(0.043)		
λ_1	2.14*	0.14	-0.42	0.33
	(0.25)	(1.17)	(2.16)	(3.41)
	-0.32	-0.28	-0.61	-0.45
	(1.15)	(5.99)	(10.01)	(15.89)
	-0.37	-0.11	2.05	-0.29
	(0.69)	(3.74)	(5.99)	(9.52)
	-0.38	0.65	-1.13	-0.59
	(0.69)	(3.61)	(6.00)	(9.53)

Notes: Standard errors appear in parentheses. An asterisk indicates significance at the 5 per cent level.

Table 6. Parameter Estimates: Equilibrium Model

Parameter	Estimate	Standard error	t-statistic
Model without regimes			
Standard expected utility case			
δ	0.95	0.013	73.26
$\gamma (= \varphi)$	4.76	0.52	7.29
Reference level model			
δ	0.99	0.0032	316.87
γ	2.73	0.11	15.48
φ	2.13	0.12	9.77
κ	0.60	0.011	53.45
b_0	0.031	0.0076	4.06
b_1	-1.74	0.079	-21.45
b_2	0.15	0.042	3.70
b_3	0.91	0.43	2.09
b_4	0.53	0.54	0.98
Model with regimes			
δ	0.99	0.0062	159.58
γ	1.11	0.27	0.41
φ	1.13	0.26	0.48
κ	-0.021	0.0009	-23.67
b_0	0.88	0.29	2.95
b_1	29.80	3.54	8.40
b_2	0.88	2.21	0.39
b_3	-17.03	24.66	-0.69
b_4	-11.01	25.68	-0.42

Notes: The t-statistics for γ and φ are for the null hypothesis that the parameter equals one. The other t-statistics in the table are for the null hypothesis that the corresponding parameter equals zero.

Table 7. Absolute Pricing Errors (Percentage Points)

	Maturity in quarters					
	2	4	8	12	16	20
No-arbitrage model						
Mean	0.21	0.38	0.52	0.56	0.60	0.69
Std. dev.	0.22	0.36	0.47	0.48	0.50	0.51
Min	0	0.02	0	0	0	0.08
Max	1.05	1.56	2.02	1.96	2.09	2.22
Equilibrium model without regimes						
Standard expected utility case						
Mean	3.35	3.01	2.72	2.59	2.55	2.53
Std. dev.	2.13	2.07	2.08	2.06	2.05	2.06
Min	0.04	0.05	0.05	0.02	0.02	0
Max	8.63	9.14	8.61	8.06	7.70	7.45
Reference level model						
Mean	0.54	0.48	0.51	0.56	0.65	0.77
Std. dev.	0.39	0.35	0.41	0.48	0.53	0.56
Min	0.01	0	0.02	0	0	0.02
Max	1.70	1.33	1.72	2.05	2.23	2.38
Equilibrium model with regimes						
Mean	0.95	0.84	0.82	0.96	1.16	1.40
Std. dev.	0.62	0.48	0.61	0.93	1.29	1.62
Min	0.01	0.03	0.03	0.02	0.02	0
Max	2.54	2.08	2.27	3.23	4.22	5.14

Note: Values less than 10^{-2} are reported as zero.

Table 8. One-Quarter-Ahead Absolute Forecast Errors (Percentage Points)

	Maturity in quarters					
	2	4	8	12	16	20
No-arbitrage model						
Mean	0.51	0.55	0.72	0.85	0.95	1.02
Std. dev.	0.42	0.46	0.48	0.49	0.53	0.60
Min	0.02	0.01	0	0.02	0.03	0.04
Max	1.83	2.04	2.07	2.02	2.46	2.88
Equilibrium model without regimes						
Mean	1.13	0.88	0.65	0.57	0.55	0.54
Std. dev.	0.49	0.42	0.47	0.48	0.48	0.50
Min	0.08	0.14	0	0	0.02	0.03
Max	2.15	1.92	2.02	2.12	2.11	2.08

Note: Values less than 10^{-2} are reported as zero.

Figure 1. Canadian GDP (top panel) and current depth of recession (bottom panel).

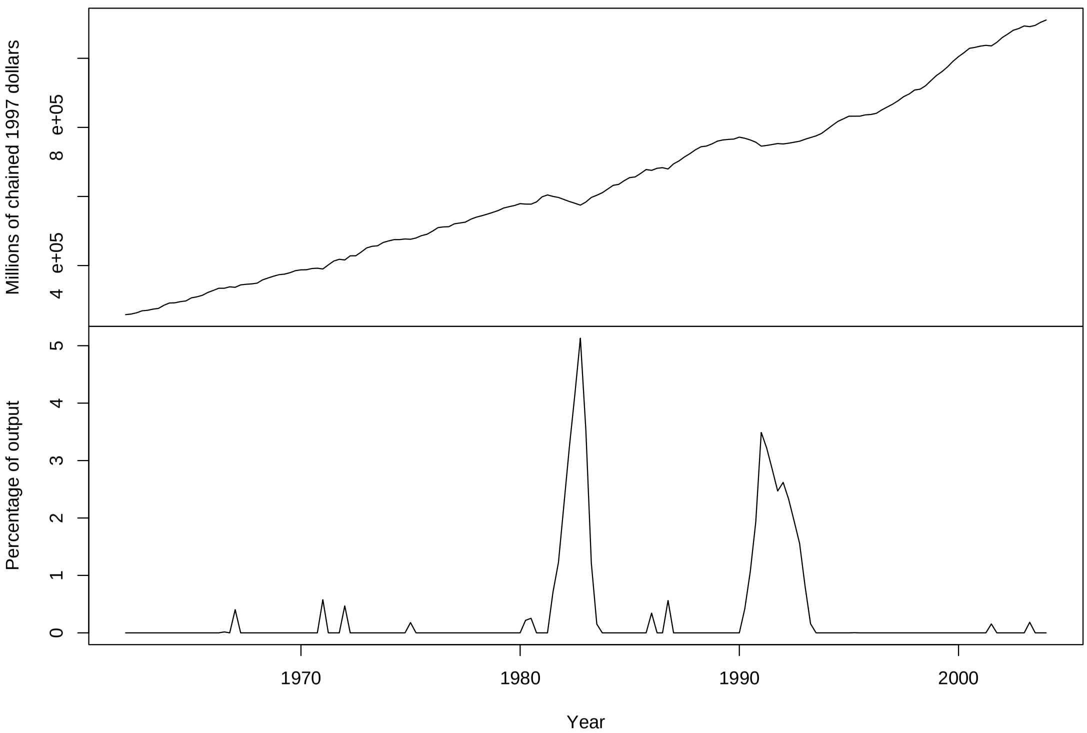


Figure 2. Recession indicator (top panel) and probabilities of recession (bottom panel) from the preferred probit model specification.

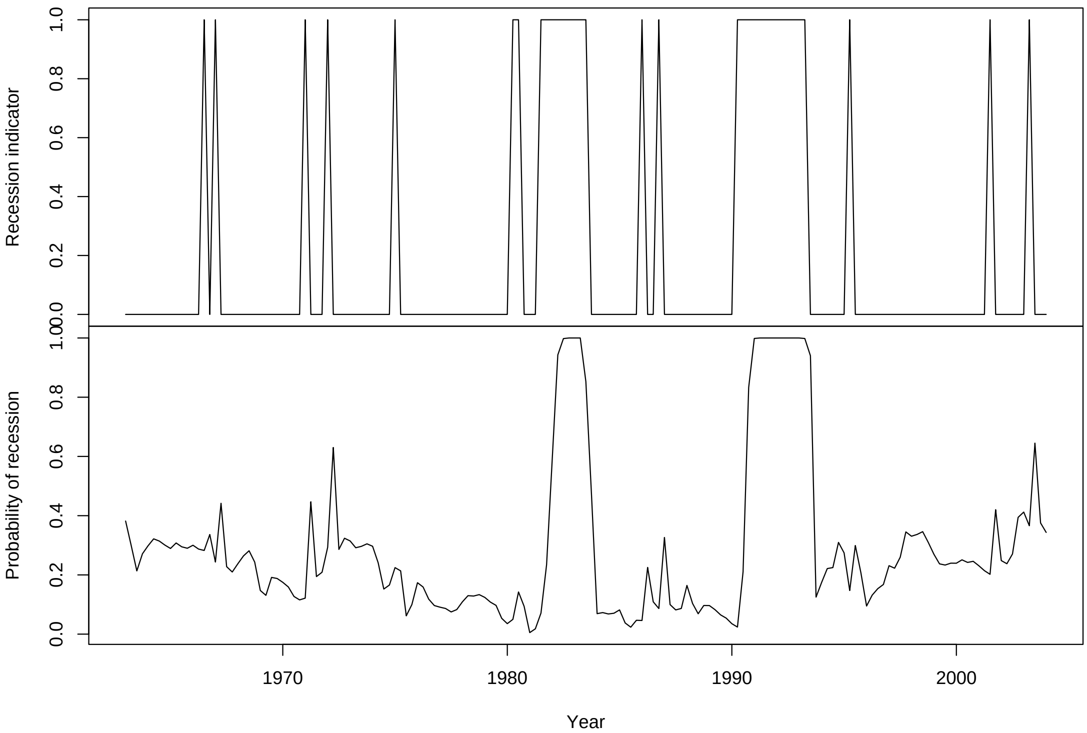


Figure 3. The solid, dashed, and dotted lines represent the 2-, 8-, and 20-quarter yields, respectively.

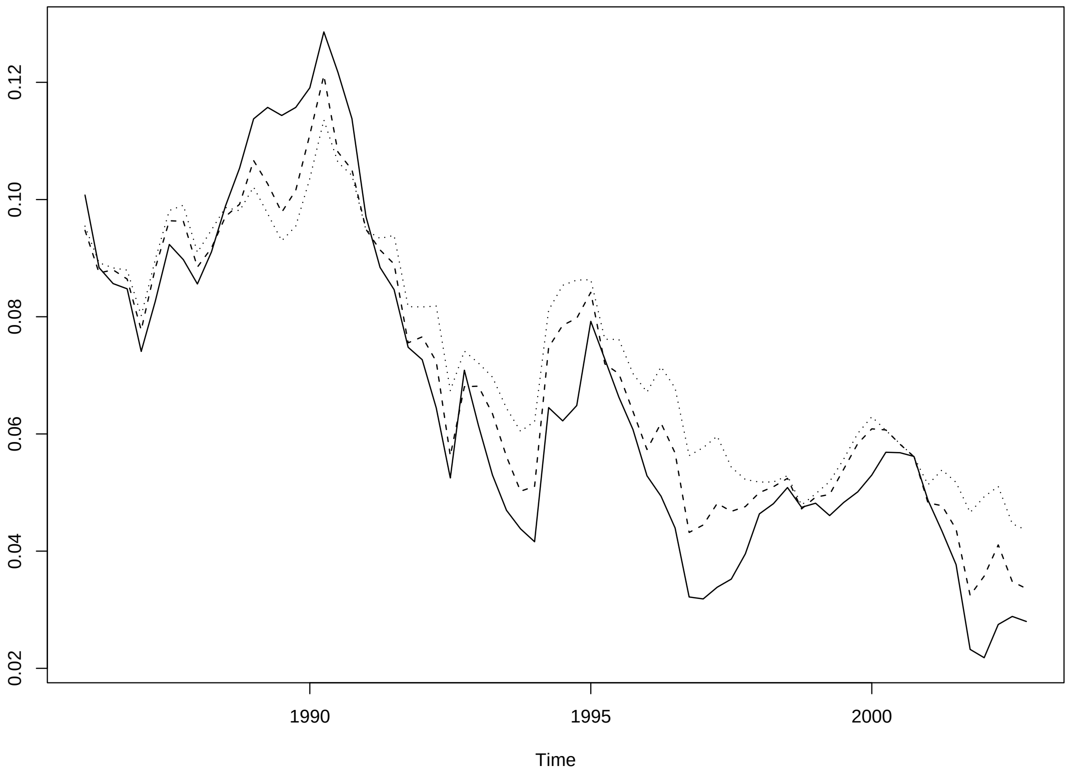
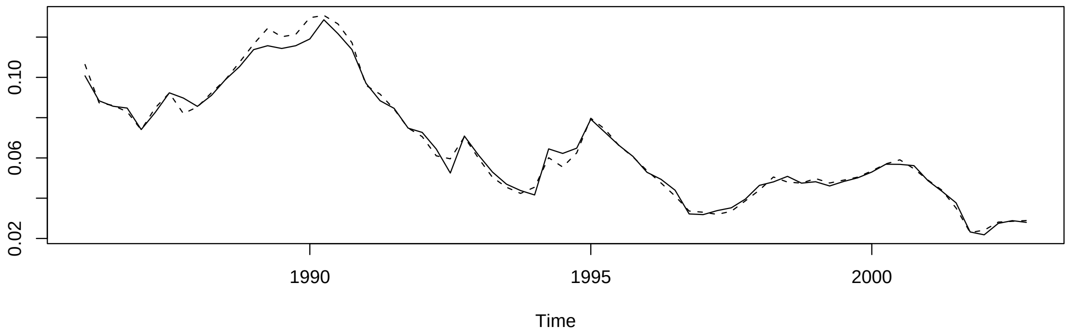


Figure 4. The solid and dashed lines represent the actual and fitted 2-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.



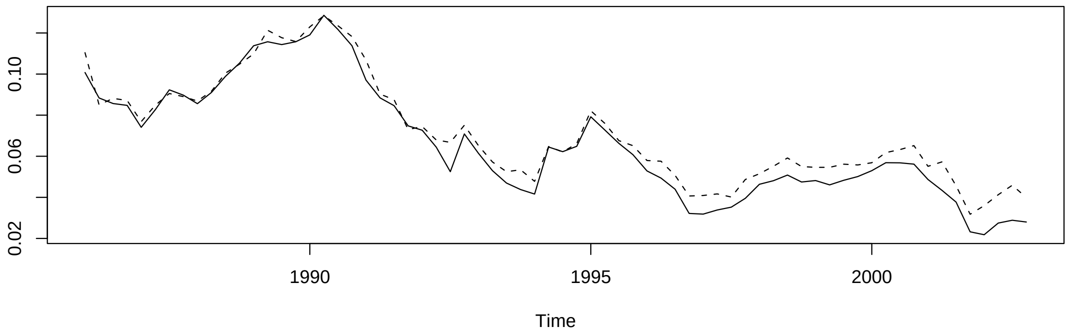


Figure 5. The solid and dashed lines represent the actual and fitted 8-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.

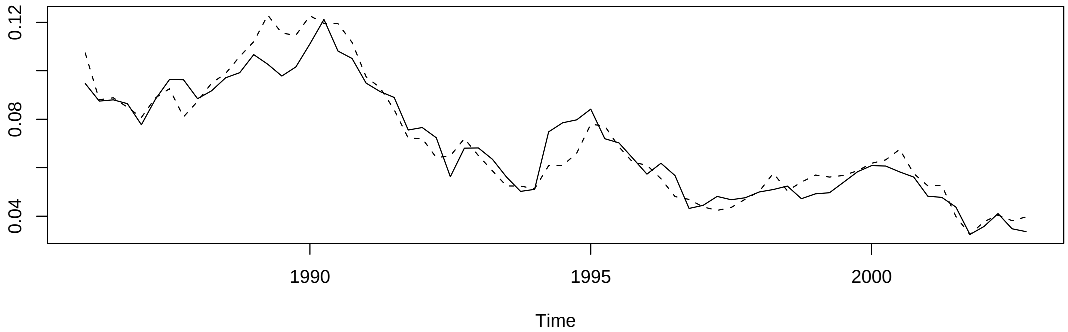


Figure 6. The solid and dashed lines represent the actual and fitted 20-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.

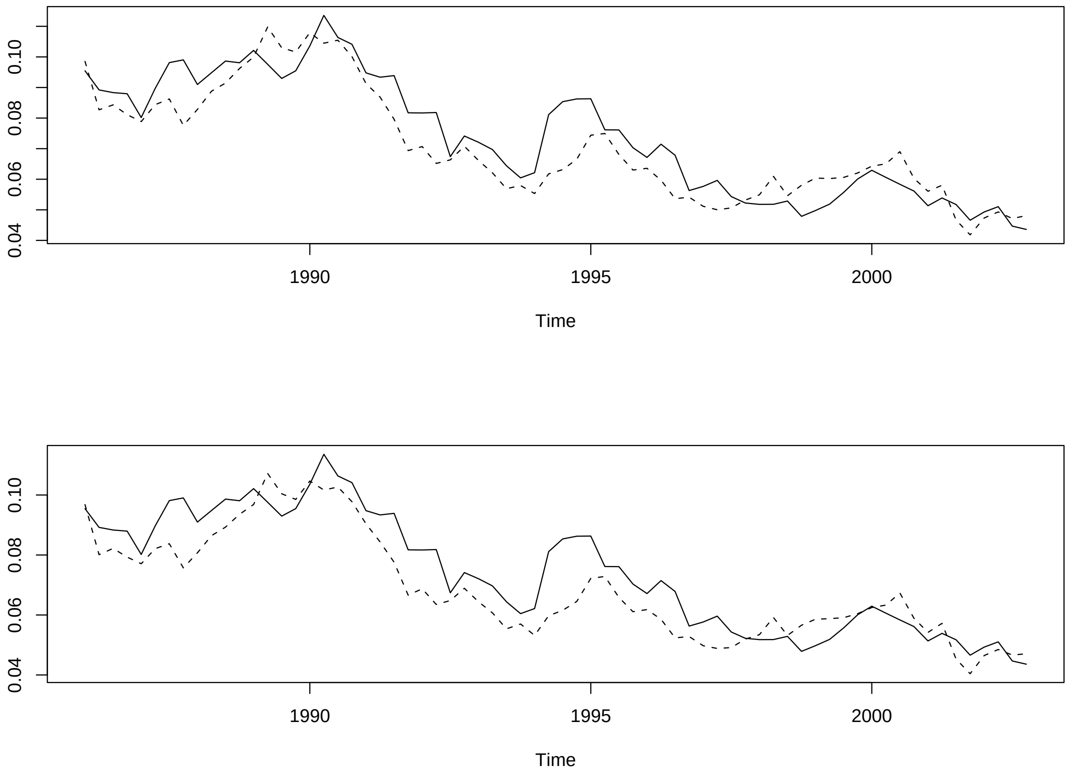


Figure 7. The solid and dashed lines represent the actual and one-quarter-ahead predicted 2-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.

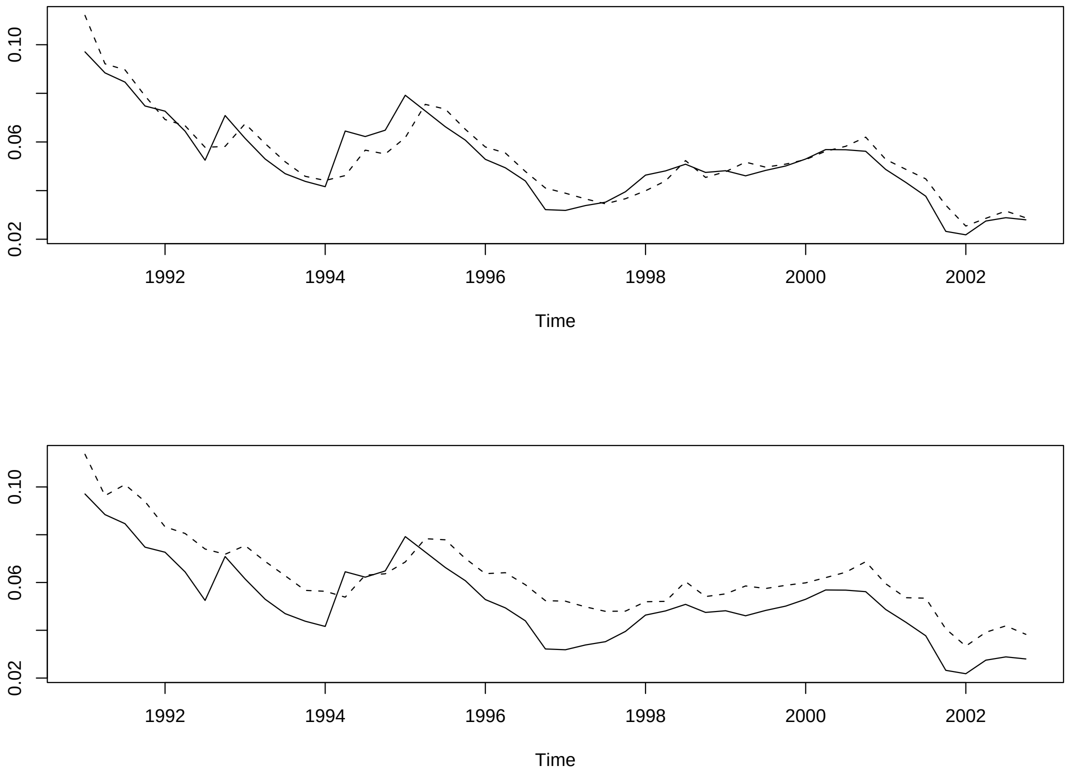


Figure 8. The solid and dashed lines represent the actual and one-quarter-ahead predicted 8-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.

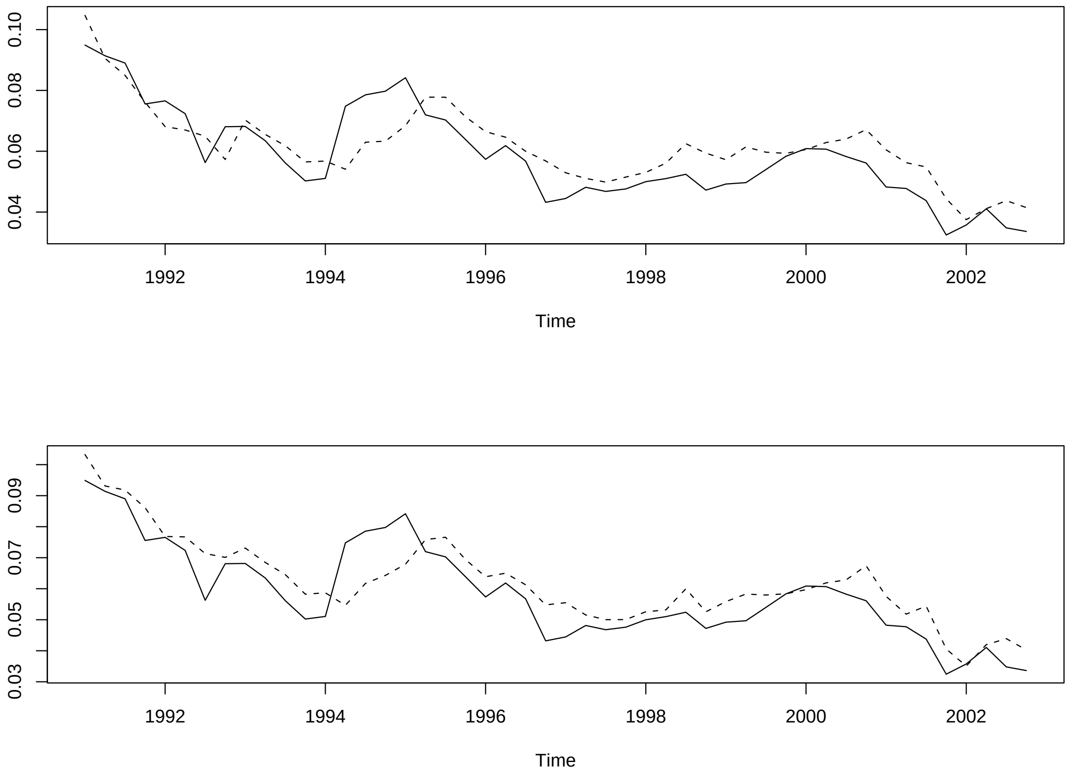


Figure 9. The solid and dashed lines represent the actual and one-quarter-ahead predicted 20-quarter yields. The top panel represents the no-arbitrage model, and the bottom panel represents the equilibrium model without regimes.

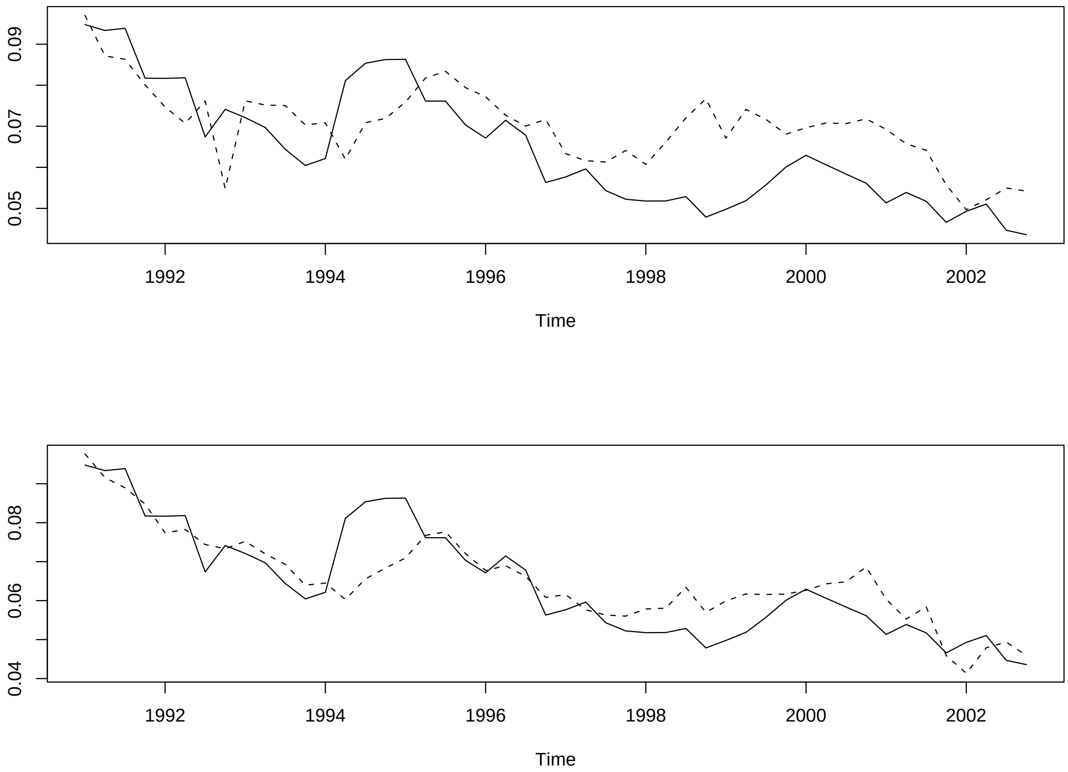


Figure 10. Impulse responses of 2-quarter yield.

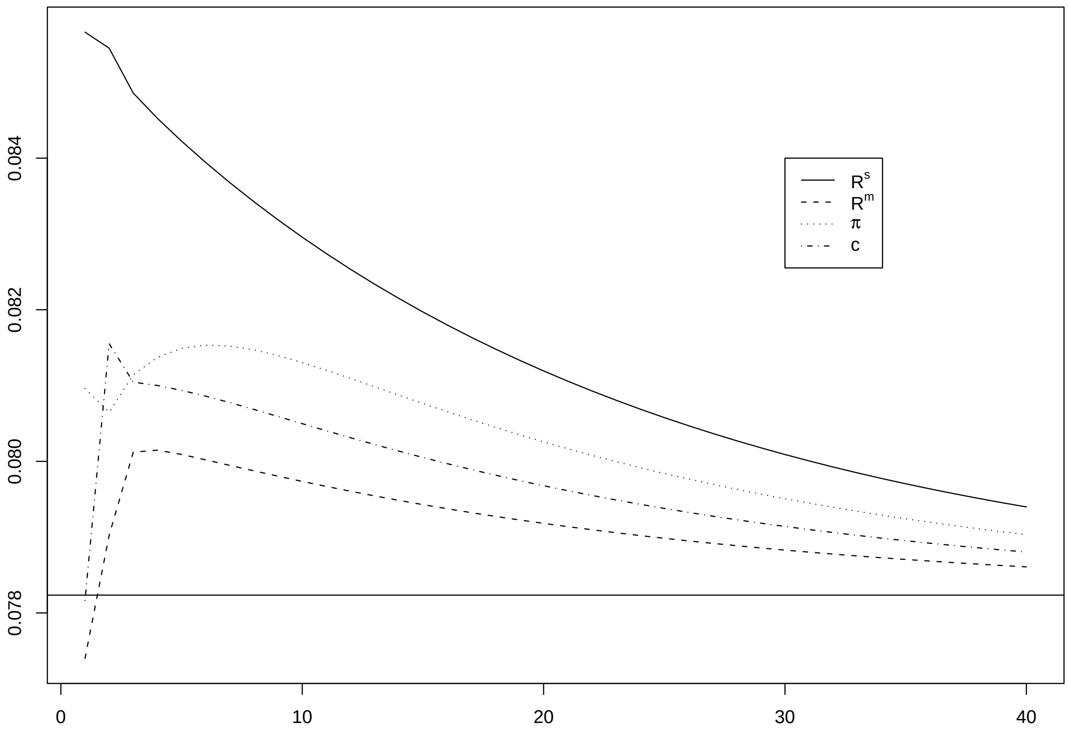


Figure 11. Impulse responses of 20-quarter yield.

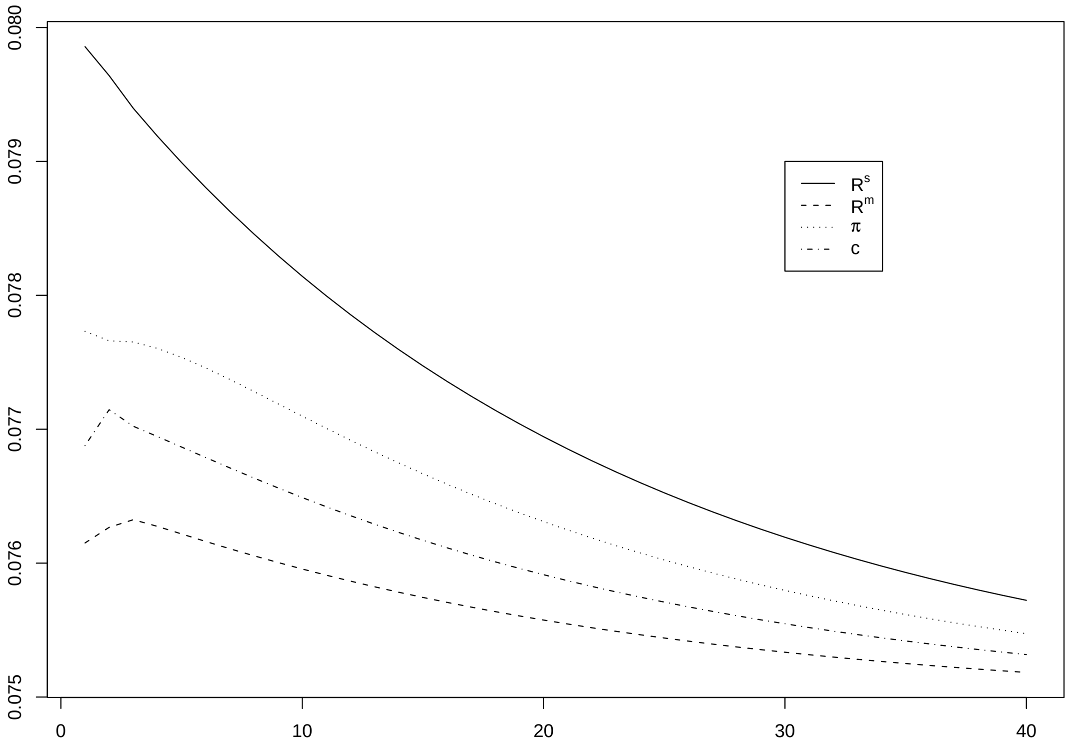
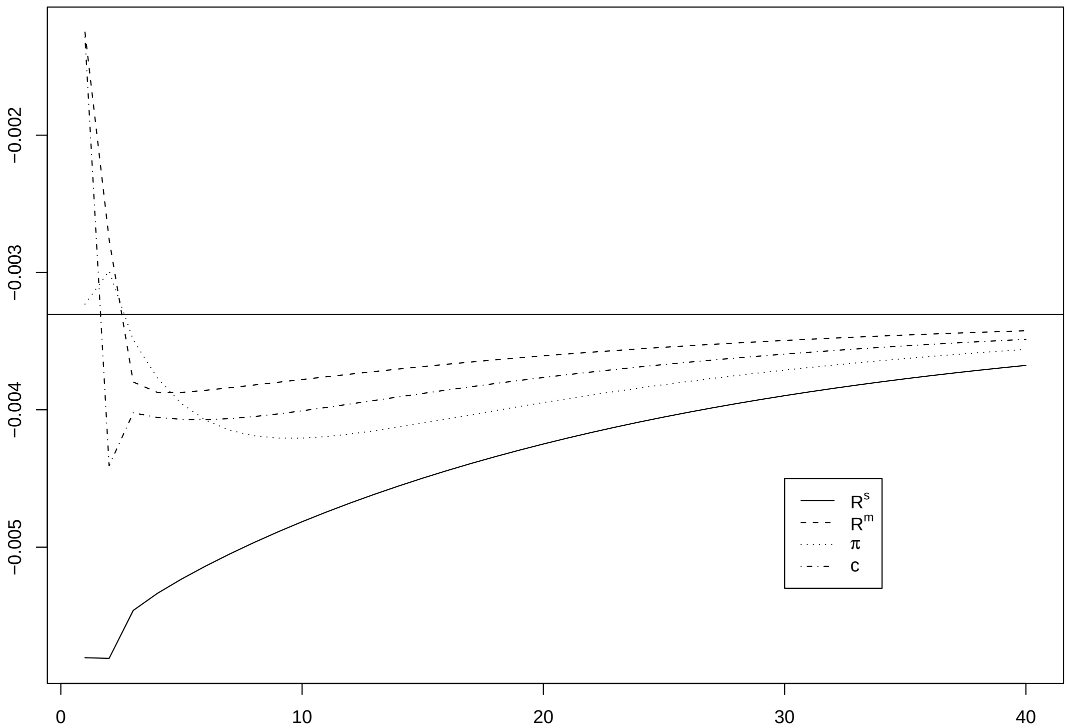


Figure 12. Impulse responses of spread between 20- and 2-quarter yields.



Appendix: VAR Estimation Results

This appendix provides the estimation results for the considered VAR models. Standard errors are shown in parentheses.

Model without regimes

$$\hat{\mu} = \begin{bmatrix} -0.0046 \\ (0.0020) \\ 0.029 \\ (0.016) \\ -0.0013 \\ (0.0011) \\ 0.016 \\ (0.0017) \end{bmatrix}, \quad \hat{\Phi} = \begin{bmatrix} 0.97 & 0.017 & 0.23 & 0.42 \\ (0.025) & (0.0097) & (0.101) & (0.086) \\ -0.20 & 0.19 & 0.23 & -0.56 \\ (0.20) & (0.077) & (0.80) & (0.68) \\ 0.043 & 0.0013 & 0.72 & 0.17 \\ (0.014) & (0.0053) & (0.056) & (0.047) \\ -0.120 & 0.028 & 0.16 & -0.14 \\ (0.022) & (0.0084) & (0.087) & (0.074) \end{bmatrix},$$

$$\text{vech}(\hat{\Omega}) = \begin{bmatrix} 7.79 \times 10^{-5} \\ (8.61 \times 10^{-6}) \\ -1.12 \times 10^{-4} \\ (4.91 \times 10^{-5}) \\ 4.90 \times 10^{-3} \\ (5.41 \times 10^{-4}) \\ 6.27 \times 10^{-6} \\ (3.38 \times 10^{-6}) \\ -3.10 \times 10^{-5} \\ (2.65 \times 10^{-5}) \\ 2.34 \times 10^{-5} \\ (2.59 \times 10^{-6}) \\ -2.66 \times 10^{-7} \\ (5.28 \times 10^{-6}) \\ 2.67 \times 10^{-5} \\ (4.18 \times 10^{-5}) \\ -4.36 \times 10^{-6} \\ (2.91 \times 10^{-6}) \\ 5.85 \times 10^{-5} \\ (6.46 \times 10^{-6}) \end{bmatrix}$$

Model with regimes

$$\hat{\mu}_0 = \begin{bmatrix} -0.0039 \\ (0.0019) \\ 0.027 \\ (0.015) \\ -0.0013 \\ (0.0011) \\ 0.016 \\ (0.0016) \end{bmatrix}, \qquad \hat{\mu}_1 = \begin{bmatrix} -0.0097 \\ (0.0026) \\ 0.043 \\ (0.021) \\ -0.0014 \\ (0.0015) \\ 0.011 \\ (0.0022) \end{bmatrix},$$

$$\hat{\Phi} = \begin{bmatrix} 0.99 & 0.014 & 0.19 & 0.35 \\ (0.025) & (0.0095) & (0.099) & (0.086) \\ -0.26 & 0.19 & 0.33 & -0.36 \\ (0.21) & (0.077) & (0.81) & (0.70) \\ 0.043 & 0.0012 & 0.72 & 0.17 \\ (0.014) & (0.0053) & (0.056) & (0.048) \\ -0.097 & 0.025 & 0.12 & -0.20 \\ (0.022) & (0.0082) & (0.085) & (0.074) \end{bmatrix}$$

$$\text{vech}(\hat{\Omega}_0) = \begin{bmatrix} 5.46 \times 10^{-5} \\ (6.77 \times 10^{-6}) \\ -3.68 \times 10^{-5} \\ (4.38 \times 10^{-6}) \\ 4.54 \times 10^{-3} \\ (5.63 \times 10^{-4}) \\ 4.55 \times 10^{-6} \\ (3.08 \times 10^{-6}) \\ -3.34 \times 10^{-5} \\ (2.81 \times 10^{-5}) \\ 2.23 \times 10^{-5} \\ (2.76 \times 10^{-6}) \\ -4.31 \times 10^{-6} \\ (4.76 \times 10^{-6}) \\ 2.30 \times 10^{-5} \\ (4.33 \times 10^{-5}) \\ -2.89 \times 10^{-6} \\ (3.04 \times 10^{-6}) \\ 5.36 \times 10^{-5} \\ (6.65 \times 10^{-6}) \end{bmatrix}, \qquad \text{vech}(\hat{\Omega}_1) = \begin{bmatrix} 1.47 \times 10^{-4} \\ (3.57 \times 10^{-5}) \\ -3.47 \times 10^{-4} \\ (1.73 \times 10^{-4}) \\ 6.10 \times 10^{-3} \\ (1.48 \times 10^{-3}) \\ 1.25 \times 10^{-5} \\ (1.12 \times 10^{-5}) \\ -2.10 \times 10^{-5} \\ (7.07 \times 10^{-5}) \\ 2.78 \times 10^{-5} \\ (6.74 \times 10^{-6}) \\ -4.06 \times 10^{-6} \\ (1.59 \times 10^{-5}) \\ 9.37 \times 10^{-5} \\ (1.03 \times 10^{-4}) \\ -1.02 \times 10^{-5} \\ (7.13 \times 10^{-6}) \\ 5.84 \times 10^{-5} \\ (1.41 \times 10^{-5}) \end{bmatrix}$$