State Dependence Explains Risk Aversion Puzzle

Abstract

This paper provides a unifying framework to consider theoretical and statistical explanations to the risk aversion puzzle put forward in Aït-Sahalia and Lo (2000) and Jackwerth (2000). We consider a setting where the stochastic discount factor and the returns on a market index are jointly dependent upon a vector of latent state variables. Lack of observability of these state variables creates conditioning issues that rationalize the risk aversion puzzles in the sense that, once properly conditioned, the pricing kernel and risk aversion functions are consistent with economic theory. Without the conditioning information, the objective distribution acquires fatter tails, while the risk neutral distribution becomes more negatively skewed. As a result, a risk aversion puzzle is inferred. To differentiate between the various theoretical explanations in terms of heterogeneity of beliefs or preferences, market sentiment, state-dependent utility or regimes in fundamentals, we propose several consumption-based asset pricing models. Based on a reasonable calibration of these models, we show that they can reproduce the risk aversion and pricing kernel puzzles found in the data.

Keywords: Risk Aversion Puzzle, State Dependence, Option Pricing, Heterogeneity in Beliefs.

JEL classification: G12, G13.

1. Introduction

Several researchers have extracted risk aversion functions or preference parameters from observed stock and bond prices. In a complete market economy, which implies the existence of a representative investor, absolute risk aversion can be evaluated for any state of wealth in terms of the historical and risk neutral distributions. Aït-Sahalia and Lo (2000) and Jackwerth (2000) have proposed nonparametric approaches to recover risk aversion functions across wealth states from observed stock and option prices. Rosenberg and Engle (2002), Garcia, Luger and Renault (2003) and Bliss and Panigirtzoglou (2004) have estimated preference parameters based on parametric asset pricing models with several specifications of the utility function.

These efforts to exploit prices of financial assets to recover fundamental economic parameters have produced puzzling results. Aït-Sahalia and Lo (2000) find that the nonparametrically implied function of relative risk aversion varies significantly across the range of S&P 500 index values, from 1 to 60, and is U-shaped. Jackwerth (2000) finds also that the implied absolute risk aversion function is U-shaped around the current forward price but even that it can become negative. Parametric empirical estimates of the coefficient of relative risk aversion also show considerable variation. Rosenberg and Engle (2002) report values ranging from 2.36 to 12.55 for a power utility pricing kernel across time[1], while Bliss and Panigirtzoglou (2004) estimate average values between 2.33 and 11.14 for the same S&P 500 index for several option maturities. Garcia, Luger and Renault (2003) estimate a consumption-based asset pricing model with regime-switching fundamentals and Epstein and Zin (1989) preferences. The estimated parameters for risk aversion and intertemporal substitution are reasonable, with average values of 0.6838 and 0.8532 respectively over the 1991-1995 period [2].

The main goal of this paper is to reconcile these facts with economic theory. Our central contribution will be to provide a unifying explanation for the various reported results. While several statistical, behavioral and theoretical explanations have been proposed, they lack a common thread to understand the fundamental reason why these results are obtained. We argue that state dependence lies at the heart of the nonparametric risk aversion puzzles or the variability of the parametric estimates. State dependence refers to a conditioning set of state variables, typically unobservable either to the researcher or the investors or both. By assuming joint lognormality for the returns on a stock index and a general stochastic discount factor, conditionally to this set of state variables, we derive an option pricing formula that admits as special cases several option pricing formulas based on equilibrium or absence of arbitrage. To obtain the risk neutral distribution we appeal to Breeden and Litzenberger (1978) formula relating the second derivative of the option pricing formula to the risk neutral distribution.

The conditioning set differs whether we consider the future state variables or the past state variables. We argue that these conditioning issues rationalize the risk aversion puzzles in the sense that, once properly conditioned, the pricing kernel and risk aversion functions are consistent with economic theory. Without the conditioning information, the objective distribution acquires fatter tails, while the risk neutral distribution becomes more negatively skewed. As a result, a risk aversion puzzle is inferred.

Two main theoretical explanations have been put forth to explain the risk aversion puzzle. Ziegler (2002) argues that heterogeneity of beliefs is the most likely cause of the U-shaped pattern. He writes a simple model with two types of CARA agents with weights ϕ and $1-\phi$ in the economy. Both types view future asset prices as lognormally distributed, but differ in their estimates of the mean and variance of the process. The estimates suggest that there are two regimes but the interpretation that beliefs are heterogeneous may not be warranted[3]. The mixture could be interpreted instead as two regimes in the fundamentals and the parameter ϕ as the probability of being in one regime. Brown and Jackwerth (2000)[4] assume instead that the representative investor has state-dependent utility, which they interpret as generalized utility (for example external habit formation as in Campbell and Cochrane, 1999) or simply a utility with volatility as an additional state variable. They retain the latter to construct a pricing kernel that is a weighted average of two pricing kernels associated with high and low volatility. Again this high and low volatility could be associated with two volatility regimes for the endowment process in the economy.

Behavioral explanations have also been put forward, mainly by Shefrin (2000, 2005), but amount to a model with heterogeneous beliefs similar to Ziegler (2002) with two types of investors. More convincingly, Han (2004) establishes an empirical link between the risk-neutral skewness of index return and investor sentiment measured by an institutional investor index. Since the index risk-neutral skewness is related to the slope of the pricing kernel with respect to the index return, market sentiment seems to affect the pricing kernel.

Another contribution of the paper is to propose two consumption-based asset pricing models with state variables that can help sort out the various hypotheses about the source of the puzzle. The models will permit to determine if state dependence manifests itself in preferences, economic fundamentals or beliefs. A first model generalizes two existing models based on recursive utility. Garcia, Luger and Renault (2001) proposed a general asset pricing model with Epstein and Zin (1989) preferences where the joint process of consumption and dividends is lognormal conditionally to a set of state variables[5]. In Melino and Yang (2003), investors' preferences are also based on the Epstein and Zin model but they are state dependent. We derive an option pricing formula in closed-form for this generalized Epstein-Zin asset pricing model and characterize analytically the risk neutral and objective distributions based on several assumptions about the conditioning information. We also derive the two distributions and a corresponding option pricing formula in a model with belief-dependent utility with external habit formation proposed by Veronesi (2004) where the uncertainty is again characterized by a set of state variables.

To illustrate the risk aversion puzzles we assume that these state variables follow a discrete-state Markov process with two states. In these models, by construction, the risk aversion functions are consistent with economic theory within each regime since the regime is observed by investors. However, as in Jackwerth (2000), we obtain negative estimates of the risk aversion function in some states of wealth. The pricing kernel function across wealth states also exhibits a puzzle even though this function is decreasing within each regime.

In the discussion so far, we have left all estimation issues aside. Indeed, the puzzle comes from the fact that the states are unobservable at least by the researcher if not by investors. Several statistical methodologies are possible to recover the objective distribution of future returns (on the underlying index) given current ones. As emphasized by Jackwerth (2000), the choice of a particular estimation strategy should not have any impact on the documented puzzles. For instance, a kernel estimation will be valid under very general stationarity and mixing conditions. While historical probabilities p_j are recovered from a time series of underlying index returns, risk neutral probabilities p_j^* will be backed out in cross-sections from a set of observed option prices written on the same index. In Jackwerth (2000), the risk neutral distribution is recovered from prices on European call options written on the S&P500 index by applying a variation of the nonparametric method introduced in Jackwerth and Rubinstein (1996). The basic idea of this method is to search for the smoothest risk-neutral distribution according to a specific objective function, which at the same time explains the option prices. However, Jackwerth and Rubinstein (1996) observe that the implied distributions are rather independent of the choice of the objective function when a sufficiently high number of options is available.[6]

Finally, Brown and Jackwerth (2000) put aside other possible market explanations for the puzzles such as lack of liquidity of out-of-the-money options, market frictions due to margin account requirements, or the difficulty of hedging when selling out-of-the-money puts.

The remainder of this paper is organized as follows. In section 2, we analyze the pricing kernel and risk aversion puzzles. In section 3, we present a general asset pricing model with state variables and derive a formula for the price of a European call option on a stock index. We further characterize the objective and risk neutral probabilities across wealth states in this model. Section 4 introduces two economic models, where economic fundamentals, preferences, and investors' beliefs can be state-dependent. Section 5 contains the empirical illustrations of the puzzles based on a calibration of these economic models. In section 6, we provide a discussion to relate our framework to other approaches found in the literature. Section 7 concludes.

2. The Pricing Kernel and Risk Aversion Puzzles

In this section, we recall the puzzles put forward by Aït-Sahalia and Lo (2000) and Jackwerth (2000).

2.1 Theoretical Underpinnings

Under very general non arbitrage conditions (Hansen and Richard (1987)), the time t price of an asset which delivers a payoff g_{t+1} at time $(t+1)$ is given by:

$$p_t = E_t \left[m_{t+1} g_{t+1} \right], \tag{1}$$

where $E_t[.]$ denotes the conditional expectation operator given investors' information at time t. Any random variable m_{t+1} conformable to (1) is called an admissible stochastic discount factor (SDF). Among the admissible SDFs, only one denoted by m^*_{t+1} is a function of available payoffs. It is the orthogonal projection of any admissible SDF on the set of payoffs. If some rational investor is able to separate its utility over current and future values of consumption:

$$U[C_t, C_{t+1}] = u(C_t) + \beta u\left(C_{t+1}\right), \tag{2}$$

the first-order condition for an optimal consumption and portfolio choice will imply that m^*_{t+1} coincides with the projection of $\beta \frac{u\prime(C_{t+1})}{u\prime(C_t)}$ on the set of payoffs. Therefore, through a convenient aggregation argument, concavity of utility functions should imply that m^*_{t+1} is decreasing in current wealth.

Moreover, as shown by Hansen and Richard (1987), no arbitrage implies almost sure positivity of m^*_{t+1}. Therefore, $m^*_{t+1}/E_t m^*_{t+1}$ can be interpreted as the density function of the risk neutral probability distribution with respect to the historical one. In case of a representative investor with preferences conformable to (2), we deduce:

$$\frac{m^*_{t+1}}{E_t m^*_{t+1}} = \frac{u'\left(C_{t+1}\right)}{E_t u'\left(C_{t+1}\right)}.$$

Therefore:

$$\frac{\partial Log m^*_{t+1}}{\partial C_{t+1}} = \frac{u''\left(C_{t+1}\right)}{u'\left(C_{t+1}\right)} \tag{3}$$

is the opposite of the Arrow-Pratt index of absolute risk aversion (ARA) of the investor.

2.2 The Puzzles

For sake of simplicity, it is convenient to analyze these puzzles in a finite state space framework. If $j = 1, \cdots, n$ denote the possible states of nature, we get the density function of the risk neutral distribution probability with respect to the historical one as:

$$\frac{m_{t+1}^*}{E_t m_{t+1}^*} = \frac{p_j^*}{p_j} \text{ in state } j, \tag{4}$$

where p_j^* is the risk neutral probability across wealth states $j = 1, \cdots, n$ and p_j is the corresponding historical probability. Brown and Jackwerth (2000) use formula (4) to empirically derive the pricing kernel function from realized returns on the SNP 500 index and option prices on the index over a post-1987 period. For the center wealth states (over the range of 0.97 to 1.03 with wealth normalized to one), they found a pricing kernel function which is increasing in wealth. This is the so-called pricing kernel puzzle.

As explained in section 2.1 above, the increasing nature of the pricing kernel function in wealth is puzzling because it is akin to a convex utility function for a representative investor, which is obviously inconsistent with the general assumption of risk aversion. From (3), the ARA coefficient can actually be computed through a log-derivative of the pricing kernel. By using (4) we deduce:

$$ARA = -\frac{u''(C_{t+1})}{u'(C_{t+1})} = \frac{p_j'}{p_j} - \frac{p_j^{*\prime}}{p_j^*} \tag{5}$$

where p_j' and $p_j^{*\prime}$ are of the derivatives of p_j and p_j^* with respect to aggregate wealth in state j.

Jackwerth (2000) observes that the absolute risk aversion functions computed from (5) dramatically change shape around the 1987 crash. Prior to the crash, they are positive and decreasing in wealth, which is consistent with standard assumptions made in economic theory about investors' preferences. After the crash, they are partially negative and increasing (see figure 3 in Jackwerth (2000)). This result is called the risk aversion puzzle. One component of it is equivalent to the pricing kernel puzzle: ARA should be positive as the pricing kernel should be decreasing in aggregate wealth. Additionally, even when there is no pricing kernel puzzle (positive ARA), there remains a risk aversion puzzle when ARA is increasing in wealth. While the pricing kernel puzzle is only observed for the center of wealth states, the risk aversion puzzle (increasing ARA) remains for larger levels of wealth. Without any discretization of wealth states, Aït-Sahalia and Lo (2000) documented similar empirical puzzles for implied risk aversion.

3. A General Asset Pricing Model with State Variables

In this section, we will specify a general dynamic asset pricing model with latent variables. This model will allow us to extract analytically the risk-neutral and the historical probabilities (p_j^* and p_j respectively) across wealth states. Therefore, we will be able to deduce the absolute risk aversion function as well as the pricing kernel function. To obtain the risk neutral distribution we will appeal to Breeden and Litzenberger (1978) formula relating the second derivative of the option pricing formula to the risk neutral distribution. Therefore, we will first develop an option pricing formula in this general dynamic asset pricing framework and then provide expressions for the historical and risk neutral distributions.

In order to specify a dynamic asset pricing model in discrete time, our focus of interest will be the dynamic properties of a positive stochastic discount factor (SDF) denoted by $m_{t,T}$. Since agents observe typically more than the econometrician, the information set I_t at time t may contain not only past values of prices and payoffs, but also some latent state variables.

Extending the Hansen and Richard (1987) setting to an intertemporal framework and applying the law of iterated expectations, the log-SDFs necessarily fulfill:

$$\log m_{t,T_2} = \log m_{t,T_1} + \log m_{T_1,T_2} \text{ , for } t < T_1 < T_2, \tag{6}$$

and therefore: $m_{t,T} = \prod_{\tau=t}^{T-1} m_\tau$, with: $m_\tau = m_{\tau-1,\tau}$. To show the generality of the latent state approach, we directly specify the time-series properties of the stochastic process m_t, $t = 1, 2, ..T$.

3.1 An Option Pricing Formula with State Variables

The key feature of this asset pricing model is an assumption about the sequence $(m_\tau)_{1\leq\tau\leq T}$ of unit period SDFs which amounts to a factor structure in the longitudinal dimension. A number of state variables summarize the stochastic dependence of the consecutive SDFs, in the sense that, given the state variables, they are mutually conditionally independent. The same assumption is made about the sequence of consecutive returns of the primitive asset of interest on which options are written. Therefore, in terms of the joint distribution of m_t and returns on a given asset price S_t, we maintain the following assumption[7].

Assumption 1: *The variables $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{1\leq\tau\leq T-1}$ are conditionally serially independent given the path $U_1^T = (U_t)_{1\leq t\leq T}$ of a vector U_t of state variables.*

Assumption 1 expresses that the dynamics of the returns is driven by the state variables. A similar assumption is made in common stochastic volatility models (the stochastic volatility process being the state variable) when standardized returns are assumed to be independent.

Assumption 2: *The process $\left(m_t, \frac{S_t}{S_{t-1}}\right)$ does not Granger-cause the state variables process (U_t).*

This assumption states that the state variables are exogenous. For common stochastic volatility or hidden Marlov processes, such an exogeneity assumption is usually maintained to make the standard filtering strategies valid. It should be noted that this exogeneity assumption does not preclude instantaneous causality relationships such as a leverage effect.

Assumption 3 : *The conditional probability distribution of* $\left(\log m_{t+1}, \log \frac{S_{+1}}{S_t}\right)$ *given* U_1^{t+1} *is a bivariate normal;*

$$\begin{bmatrix} \log\ m_{t+1} \\ \log \dfrac{S_{t+1}}{S_t} \end{bmatrix} |U_1^{t+1} \rightsquigarrow \mathcal{N}\left[\begin{pmatrix} \mu_{mt} \\ \mu_{st} \end{pmatrix}, \begin{pmatrix} \sigma_{mt}^2 & \sigma_{mst} \\ \sigma_{mst} & \sigma_{st}^2 \end{pmatrix}\right].$$

Assumption 3 is a very general version of the mixture of normals model. A maintained assumption will be that investors observe U_t at time t, so that the conditioning information in the expectation operator (3.1) is:

$$I_t = \sigma\left[m_\tau, S_\tau, U_\tau, \tau \leq t\right]. \tag{7}$$

Proposition 3.1 *Under assumptions 1, 2 and 3, the price of a European call option* π_t, *is given by:*

$$\frac{\pi_t}{S_t} = \pi_t(x_t) = E_t\left\{Q_{ms}(t,T)\,\Phi\left(d_1(x_t)\right) - \frac{\tilde{B}(t,T)}{B(t,T)} e^{-x_t}\Phi\left(d_2(x_t)\right)\right\},$$

where $x_t = \log \frac{S_t}{KB(t,T)}$, $B(t,T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T, and*

$$d_1(x) = \frac{x}{\bar{\sigma}_{t,T}} + \frac{\bar{\sigma}_{t,T}}{2} + \frac{1}{\bar{\sigma}_{t,T}} \log\left[Q_{ms}(t,T)\frac{B(t,T)}{\tilde{B}(t,T)}\right], \quad d_2(x) = d_1(x) - \bar{\sigma}_{t,T}, \quad \bar{\sigma}_{t,T}^2 = \sum_{\tau=t}^{T-1} \sigma_{s\tau}^2$$

and

$$\begin{aligned} \tilde{B}(t,T) &= \exp\left(\sum_{\tau=t}^{T-1} \mu_{m\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma_{m\tau}^2,\right) \\ Q_{ms}(t,T) &= \tilde{B}(t,T)\exp\left(\sum_{\tau=t}^{T-1} \sigma_{ms\tau+1}\right) E\left[\frac{S_T}{S_t}|U_1^T\right]. \end{aligned}$$

PROOF. See Appendix. ∎

This formula admits as special cases several option pricing formulas based on equilibrium or absence of arbitrage [8]. Regarding the equilibrium approach, our setting is very general since it is based on a stochastic model for the SDF which does not rely on restrictive assumptions about preferences, endowments, or agent heterogeneity. Moreover, the factorization for the SDF is more general than the usual product of intertemporal marginal rates of substitution in time-separable utility models. Indeed, as we will see in the next section, the SDF allows to accommodate non-separable or state-dependent preferences or beliefs.

3.2 Characterizing the Objective and Risk Neutral Probabilities

The option pricing formula derived in the previous section allows us to recover the risk-neutral probability distribution by appealing to Breeden and Litzenberger (1978). Assumptions 1 to 3 provide us also with the elements to derive the objective probability distributions, both conditional and unconditional.

3.2.1 Objective Probabilities Across Wealth States

Under Assumption 3, given the trajectory of the state variables, the log stock return, $\log R_{t,T} = \frac{S_T + D_T}{S_t}$, is normally distributed:

$$\log R_{t,T} | U_1^T \rightsquigarrow N\left(\mu_r, \sigma_r^2\right).$$

Using the definition of a return and the other assumptions, we are able to rewrite the return as:

$$\log R_{t,T} = \log\left(\frac{S_T + D_T}{S_t}\right) = \sum_{\tau=t}^{T-1} \log \frac{\left(\varphi\left(U_1^{\tau+1}\right) + 1\right)}{\varphi\left(U_1^{\tau}\right)} + \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau},$$

where: $\frac{S_t}{D_t} = \varphi\left(U_1^t\right)$.

Therefore, the mean and variance functions of the normal are:

$$\mu_r = \sum_{\tau=t}^{T-1} \log \frac{\left(\varphi\left(U_1^{\tau+1}\right) + 1\right)}{\varphi\left(U_1^{\tau}\right)} + \sum_{\tau=t}^{T-1} \mu_{Y\tau+1} \text{ and } \sigma_r^2 = \sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2$$

To derive the objective probabilities across wealth states, we have to discretize the objective density of $R_{t,T}$ onto wealth states $\left(u^j d^{n-j}\right)_{j=0,1\ldots n}$ by letting $u = e^{\sigma\sqrt{(T-t)/n}}$, $ud = 1$, where σ is the standard deviation of returns. We denote:

$$p_j\left(U_1^T\right) = P\left(R_{t,T} = u^j d^{n-j} | U_1^T\right), \, j = 1, ..., n \tag{8}$$

the objective probabilities across wealth states conditionally to the trajectory of the state variables. This is to be contrasted with the objective probabilities that do not condition on the trajectory of the state variables:

$$p_j = E\{P\left(R_{t,T} = u^j d^{n-j} | U_1^T\right)\}, \, j = 1, ..., n \tag{9}$$

We can also compute the objective probabilities across wealth states given U_1^t, the past of the state variables.

$$p_j\left(U_t\right)=E_t p_j\left(U_1^T\right),\; j=1,...,n \tag{10}$$

where the expectation is taken with respect to U_{t+1}^T. Once these probabilities are computed, it is straightforward to deduce the marginal probabilities:

$$p_j=Ep_j\left(U_t\right),\; j=1,...,n \tag{11}$$

where the expectation is taken with respect to U_t. These two sets of probabilities differ by the conditioning information taken into account. The first set conditions on the whole trajectory of state variables, past and future. This is to compare our framework to other papers such as Brown and Jackwerth (2000) and Ziegler (2002) where they simulate future trajectories of the processes and estimate the distributions by kernel. In our setting we are able to characterize the distributions analytically.

The second set of probabilities in (10) and (11) is meant to condition on the available information at time t and to capture the potential informational gap between the econometrician or the researcher and the investor. While the latter may observe the past states U_1^t in t, the former will not have knowledge of it and will compute an unconditional probability.

3.2.2 Risk Neutral Probabilities

In a complete market, Ross (1976) shows that one can recover the risk neutral distribution from a set of European option prices. This distribution turns out to be unique. Breeden and Litzenberger (1978) gives an exact formula for the risk-neutral distribution:

$$\left(\frac{\partial^2\pi_t}{\partial^2 K}\right)_{K=x}=f^*\left(x\right)$$

where $f^*\left(x\right)$ is the risk neutral density of S_T at x. The proposed option pricing formula is of the form:

$$\pi_t=E_t C_{op}\left(U_1^T\right),$$

with

$$C_{op}\left(U_1^T\right)=S_t Q_{XY}\left(t,T\right)\Phi\left(d_1\left(x_t\right)\right)-K\widetilde{B}\left(t,T\right)\Phi\left(d_2\left(x_t\right)\right).$$

Therefore, applying the Breeden and Litzenberger (1978) formula gives:

$$\left(\frac{\partial^2\pi_t}{\partial^2 K}\right)_{K=x}=E_t\left(\frac{\partial^2 C_{op}\left(U_1^T\right)}{\partial^2 K}\right)_{K=x}.$$

We denote:

$$\frac{\partial^2 C_{op}\left(U_1^T\right)}{\partial^2 K} = f^*\left(x|U_1^T\right),$$

where $f^*\left(x|U_1^T\right)$ represents the risk neutral density of S_T given the trajectory of state variables. As a result, the risk neutral density of $R_{t,T} = \frac{S_T+D_T}{S_t}$ conditional on the trajectory of state variables can be obtained using a simple transformation of $f^*\left(.|U_1^T\right)$. To derive the risk neutral probabilities across wealth states, we have to discretize the risk neutral probability distribution of $R_{t,T} = \frac{S_T+D_T}{S_t}$ onto the same wealth space that we employed for the objective distribution. If p^* represents the risk neutral probability distribution of $R_{t,T} = \frac{S_T+D_T}{S_t}$ conditionally on the trajectory of state variables, we denote

$$p_j^*\left(U_1^T\right) = P^*\left(R_{t,T} = u^j d^{n-j}|U_1^T\right),\ j = 1, ..., n \tag{12}$$

the risk neutral probabilities across wealth states given the whole trajectory of the state variables. Similarly to the objective probabilities, we can compute

$$p_j^* = E\{P^*\left(R_{t,T} = u^j d^{n-j}|U_1^T\right)\},\ j = 1, ..., n \tag{13}$$

that do not condition on the trajectory of the state variables.

We also compute the risk-neutral probabilities given the past of the state variables U_1^t:

$$p_j^*\left(U_t\right) = E_t p_j^*\left(U_1^T\right) \tag{14}$$

The marginal risk neutral probabilities across wealth states are computed by taking the expectation of (14)

$$p_j^* = E p_j^*\left(U_t\right),\ j = 1, ..., n. \tag{15}$$

where the expectation is taken with respect to U_1^t. Again, the probabilities in (14) and (15) characterize the potential information gap between the investor and the researcher respectively.

Our main point will be to show that this state variable framework will be able to reproduce the risk aversion and pricing kernel puzzles. However to give an economic content to the source of the puzzles, one needs to specify models of the SDF that are interpretable in terms of economic fundamentals, preferences or beliefs since the few papers that have tried to understand the source of the puzzles have considered explanations along these potential lines. Therefore, we propose equilibrium models where fundamentals, preferences and beliefs may depend on the state variables.

4. Economies with Regime Shifts

In the previous section, we argued that the risk aversion puzzle could be rationalized by a general dependence of the stochastic discount factor on state variables. Some authors have proposed economic explanations for the risk aversion puzzle. For example, Ziegler (2002) and Shefrin (2000, 2005) make a case for the heterogeneity of beliefs, while Brown and Jackwerth (2000) rationalize the puzzle through state-dependent utility. However, ultimately, the puzzles are often illustrated by using a mixture of normals, which makes it difficult to identify the actual source of the puzzles. In this section, we propose two main classes of equilibrium models with dependence on state variables, one based on recursive utility, another on external habit formation. In these models, preferences, fundamentals or beliefs may change with the state variables, allowing for the potential identification of the actual source among several alternatives.

4.1 State-Dependent Preferences or Fundamentals in a Recursive Utility Framework

In order to consider economically meaningful regime shifts in the SDF, it is convenient to start from a two-factor model as produced by Epstein and Zin(1989). Their recursive utility framework leads them to the following SDF:

$$m_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\gamma}\left[\frac{1}{R_{mt+1}}\right]^{1-\gamma} \tag{16}$$

where $\rho = 1 - \dfrac{1}{\sigma}$, σ is the elasticity of intertemporal substitution and $\gamma = \dfrac{\alpha}{\rho}$ with $(1-\alpha)$ the index of relative risk aversion. With a two-state mixing variable U_{t+1}, $\log m_{t+1}$ appears as a mixture of two normal distributions in two cases. In the first case of state-dependent preferences, preference parameters are functions of U_{t+1} while in the second case, there are regime shifts in fundamentals and the joint probability distribution of $\left(\log\frac{C_{t+1}}{C_t}, \log R_{mt+1}\right)$ is a mixture of normals. The case of state-dependent preferences has been analyzed recently by Melino and Yang (2003), while Garcia, Luger and Renault (2001, 2003) focus on shifts in fundamentals.[9]

Let us first assume as Melino and Yang (2003) that the three preference parameters β, α, ρ are all state-dependent and then denoted as $\beta(U_t), \alpha(U_t)$ and $\rho(U_t)$. While these values, known by the investor at time t, define her time t utility level, she does not know at this date the next coming values $\beta(U_{t+1}), \alpha(U_{t+1})$ and $\rho(U_{t+1})$. Therefore, the resulting SDF will be more complicated than just replacing α, β and ρ in (16) by their state-dependent value. Melino and Yang (2003) show that the SDF is:

$$m_{t+1} = \left[\beta(U_t)\left(\frac{C_{t+1}}{C_t}\right)^{\rho(U_t)-\frac{\rho(U_t)}{\rho(U_{t+1})}}\right]^{\gamma(U_t)} R_{mt+1}^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-1} P_t^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-\frac{\alpha(U_t)}{\rho(U_t)}}, \tag{17}$$

where $\gamma(U_t) = \frac{\alpha(U_t)}{\rho(U_t)}$ and P_t is the time t price of the market portfolio. When $\beta(U_t)$, $\alpha(U_t)$, $\rho(U_t) = \rho(U_{t+1})$ are constants, this pricing kernel reduces to the Epstein and Zin SDF (16). By definition:

$$R_{mt+1} = \frac{P_{t+1} + C_{t+1}}{P_t},$$

while the underlying asset return is $\frac{S_{t+1}+D_{t+1}}{S_t}$. Asset prices P_t and S_t are then determined as discounted values of future payoffs by iterating on the following pricing formulas:

$$P_t = E_t \left[m_{t+1} \left(P_{t+1} + C_{t+1} \right) \right] \text{ and } S_t = E_t \left[m_{t+1} \left(S_{t+1} + D_{t+1} \right) \right]. \tag{18}$$

It can be shown that Assumptions 1 and 2 in the previous section are implied by similar assumptions stated for the joint process $\left(\frac{C_{t+1}}{C_t}, \frac{D_{t+1}}{D_t} \right)$. Assumption 3 will then also be implied by a similar assumption about fundamentals:

Assumption 3': *The conditional probability distribution of* $\left(\log \frac{C_{t+1}}{C_t}, \log \frac{D_{+1}}{D_t} \right)$ given U_1^{t+1} is a bivariate normal.

$$\begin{bmatrix} \log \frac{C_{t+1}}{C_t} \\ \log \frac{D_{+1}}{D_t} \end{bmatrix} | U_1^{t+1} \rightsquigarrow N \left[\begin{pmatrix} \mu_{X_{t+1}} \\ \mu_{Y_{t+1}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{X_{t+1}} & \sigma_{XY,t+1} \\ \sigma_{XY,t+1} & \sigma^2_{Y_{t+1}} \end{pmatrix} \right].$$

Propositions 4.1 and 4.2 below define an asset pricing model that nests the models of Melino and Yang (2003) and Garcia, Luger and Renault (2003).

Proposition 4.1 *: Under assumptions 1, 2 and 3', with* m_{t+1} *given by (17), the conditional probability distribution of* $\left(Log m_{t+1}, \log \frac{S_{t+1}}{S_t} \right)$ *given* U_1^{t+1} *is jointly normal with mean and variances defined in the Appendix.*

Proof. See Appendix. ∎

The general option pricing formula, which accommodates the case where both fundamentals and preferences change with the regime, is given in proposition 4.2 below. First, it is worth noticing that the equilibrium model characterizes the asset prices P_t and S_t as:

$$\begin{aligned} \frac{P_t}{C_t} &= \lambda \left(U_1^t \right) = E_t \left[m_{t+1} \left(1 + \lambda \left(U_1^{t+1} \right) \right) \frac{C_{t+1}}{C_t} \right], \\ \frac{S_t}{D_t} &= \varphi \left(U_1^t \right) = E_t \left[m_{t+1} \left(1 + \varphi \left(U_1^{t+1} \right) \right) \frac{D_{t+1}}{D_t} \right]. \end{aligned}$$

Proposition 4.2 *Under assumptions 1, 2, and 3', the price of a European call option on a dividend-paying stock is given by:*

$$\pi_t = E_t \left[S_t Q_{XY} \left(t, T \right) \Phi \left(d_1 \left(x_t \right) \right) - K \widetilde{B} \left(t, T \right) \Phi \left(d_2 \left(x_t \right) \right) \right],$$

where $x_t = \log \frac{S_t}{KB(t,T)}$, $B(t,T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T and:*

$$d_2(x_t) = \frac{x_t + \log\left(Q_{XY}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}{\sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}}, \quad d_1(x_t) = d_2(x_t) + \sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}$$

with:

$$\begin{aligned} Q_{XY}(t,T) &= \widetilde{B}(t,T)\,E_t\left[\frac{(S_T+D_T)}{S_t}|U_1^T\right]\exp\left[\sum_{\tau=t}^{T-1} a_{1\tau+1}\sigma_{XY,\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\sigma_{XY,\iota+1}\right)\right], \\ \widetilde{B}(t,T) &= \exp\left[\psi_1 + \sum_{\tau=t}^{T-1} a_{1\tau+1}\mu_{X\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\mu_{X\iota+1}\right) + \frac{\sigma^2_{Z_1}}{2}\right], \end{aligned}$$

and

$$\begin{aligned} a_{0\tau+1} &= \frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau), \\ a_{1\tau+1} &= \gamma(U_\tau)\left(\rho(U_\tau) - \frac{\rho(U_\tau)}{\rho(U_{\tau+1})}\right) + \left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - 1\right), \\ E_t\left[\frac{(S_T+D_T)}{S_t}|U_1^T\right] &= \exp\left[\sum_{\tau=t}^{T-1}\log\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^\tau\right)} + \sum_{\tau=t}^{T-1}\mu_{Y\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}\right] \end{aligned}$$

where:

$$\begin{aligned} \sigma^2_{Z_1} &= \sum_{\tau=t}^{T-1} a^2_{1\tau+1}\sigma^2_{X\tau+1} + \sum_{\tau=t+1}^{T-1} a^2_{0\tau+1}\sum_{\iota=t}^{\tau-1}\sigma^2_{X\iota+1} + 2\sum_{t\le i<j\le T-1} a_{0i+1}a_{0j+1}\sum_{\iota=t}^{i-1}\sigma^2_{X\iota+1}, \\ \psi_1 &= \sum_{\tau=t}^{T-1}\gamma(U_\tau)\log(\beta(U_\tau)) + \left[\sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau)\right)\right]\log(C_t) + \\ &\quad \sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - 1\right)\log\left(\frac{\lambda\left(U_1^{\tau+1}\right)+1}{\lambda\left(U_1^\tau\right)}\right) + \sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau)\right)\log\left(\lambda\left(U_1^\tau\right)\right). \end{aligned}$$

Proof. See Appendix ∎

If the preference parameters α, β, and ρ are constants, proposition 4.2 collapses to the Garcia, Luger and Renault (2003) option pricing formula. Note that the definition of $\lambda\left(U_1^{t+1}\right)$ and $\varphi\left(U_1^{t+1}\right)$ is equivalent to: $E_t Q_{XY}(t,t+1) = 1$, and $E_t\widetilde{B}(t,t+1) = B(t,t+1)$. This option pricing formula will help us compute the corresponding objective and risk-neutral probabilities as defined in the previous section.

4.2 An External Habit Model with State Dependence in Beliefs

Veronesi (2004) introduces the concept of belief-dependent utility functions. Consider an economy where $\mathcal{C}$ is a set of prizes and ϑ a set of states. Define a lottery $f\colon \vartheta \mapsto \Delta(\mathcal{C})$ assigning a probability distribution $\Delta(\mathcal{C})$ on $\mathcal{C}$ to each state $\upsilon \in \vartheta$. From theorem 1 in Veronesi (2004), there exists a state-dependent utility function $u : \mathcal{C} \times \vartheta \mapsto \mathbb{R}$ and a subjective probability function $\pi(.)$ such that:

$$U(C_t, \upsilon_t) = \sum_{\upsilon^i \in \vartheta} \pi_t(\upsilon^i)\, u(C_t, t|\upsilon_t = \upsilon^i)$$

where

$$\pi_t(\upsilon^i) = \Pr(\upsilon_t = \upsilon^i|\mathcal{I}_t)$$

represents the subjective probability that the state is υ^i at time t and $u(C_t, t|\upsilon_t)$ an intertemporal utility. $\mathcal{I}_t$ represents the investor information set. The utility function $U(C_t, \upsilon_t)$ is called a belief-dependent utility function over a prize C_t.

Veronesi (2004) further proposes a parsimonious parametrization that is consistent with the external habit model of Campbell and Cochrane (1999). The utility function is given by:

$$u(C_t, t|\upsilon_t) = e^{-\phi t}\frac{(C_t \upsilon_t)^{1-\alpha}}{1-\alpha}, \tag{19}$$

with the surplus consumption ratio $s_t = \upsilon_t = \frac{C_t - X_t}{C_t}$, X_t is a slow-moving external habit, C_t represents consumption at date t and ϕ is the subjective discount rate. Following Veronesi (2004), we assume that the surplus consumption ratio is perfectly correlated with the state υ_t. In this economy, the stochastic discount factor is:

$$m_t^T = \prod_{\tau=t}^{T-1} m_{\tau+1} \tag{20}$$

with

$$m_{\tau+1} = \frac{u_c(C_{t+1}, t+1|\upsilon_{t+1})}{u_c(C_t, t|\upsilon_t)}, \tag{21}$$

where $u_c(C_t, t|\upsilon_t)$ is the marginal utility of consumption at time t. The marginal utility of consumption can be expressed as

$$u_c(C_t, t|\upsilon_t) = e^{-\phi t} k(\upsilon_t)\, C_t^{-\alpha},$$

with $k(\upsilon_t) = \upsilon_t^{-\alpha}$. Substituting this last expression in (21) yields:

$$m_{\tau+1} = e^{-\phi}\left(\frac{k(\upsilon_{\tau+1})}{k(\upsilon_\tau)}\right)\left(\frac{C_{\tau+1}}{C_\tau}\right)^{-\alpha}.$$

We will now assume that

$$\frac{k(\upsilon_{\tau+1})}{k(\upsilon_\tau)} = e^{-\rho(U_\tau)\left(\log\left(\frac{C_{\tau+1}}{C_\tau}\right) - \log\left(\frac{C_\tau}{C_{\tau-1}}\right)\right)}, \tag{22}$$

where U_t represents the state variable defined in section 3. The state variable U_τ is different from υ_τ. The parameter $\rho(U_\tau)$ can be interpreted as the state dependent measure of aversion to belief uncertainty. Note that in Veronesi (2004), the parameter $\rho(U_\tau)$ is not state-dependent. Given specification (22), the stochastic discount factor $m_{\tau+1}$ can be written as

$$m_{\tau+1} = e^{-\phi} e^{-\rho(U_\tau)\left[\log\left(\frac{C_{\tau+1}}{C_\tau}\right) - \log\left(\frac{C_\tau}{C_{\tau-1}}\right)\right]} \left(\frac{C_{\tau+1}}{C_\tau}\right)^{-\alpha}.$$

Given Assumptions 1, 2 and 3', we derive the price of a European call option in this model.

Proposition 4.3 *Under assumptions 1, 2, and 3', the price of a European call option on a dividend-paying stock with strike K is worth:*

$$\pi_t = E_t\left[S_t Q_{XY}(t,T)\,\Phi(d_1) - K\widetilde{B}(t,T)\,\Phi(d_2)\right],$$

where $x_t = \log\frac{S_t}{KB(t,T)}$ $B(t,T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T and*

$$d_2(x_t) = \frac{x_t + \log\left(Q_{XY}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}{\sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}}, \quad d_1(x_t) = d_2(x_t) + \sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}},$$

and

$$Q_{XY}(t,T) = \widetilde{B}(t,T)\,E\left(\frac{(S_T+D_T)}{S_t}|U_1^T\right)\exp(\psi), \quad \widetilde{B}(t,T) = \exp\left[\mu_{Z_1} + \frac{\sigma^2_{Z_1}}{2}\right]$$

with:

$$E_t\left[\frac{(S_T+D_T)}{S_t}|U_1^T\right] = \exp\left[\sum_{\tau=t}^{T-1}\log\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)} + \sum_{\tau=t}^{T-1}\mu_{Y\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}\right]$$

and

$$\begin{aligned}
\mu_{Z_1} &= -\phi(T-t) - (\rho(U_{T-1})+\alpha)\,\mu_{X_T} + \sum_{\tau=t}^{T-2}\left[\rho(U_{\tau+1}) - (\rho(U_\tau)+\alpha)\right]\mu_{X_{\tau+1}} + \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right)\\
\sigma^2_{Z_1} &= (\rho(U_{T-1})+\alpha)^2\,\sigma^2_{X_T} + \sum_{\tau=t}^{T-2}\left[\rho(U_{\tau+1}) - (\rho(U_\tau)+\alpha)\right]^2\sigma^2_{X_{\tau+1}}\\
\psi &= \sum_{\tau=t}^{T-2}\left(\rho(U_{\tau+1}) - (\rho(U_\tau)+\alpha)\right)\sigma_{XY,\tau+1} - (\rho(U_{T-1})+\alpha)\,\sigma_{XY,T}
\end{aligned}$$

PROOF. See Appendix. ∎

5. Empirical Illustrations

To illustrate our economic models and their effects on the puzzles put forward by Aït-Shalia and Lo (2000) and Jackwerth (2000), we specify a two-state, first-order Markov process for the state variable U_t and consider in turn a recursive utility model with state-dependence in fundamentals and preferences and an external habit model with state dependence in beliefs. We produce graphs of the pricing kernel and the absolute risk aversion as functions of discrete wealth states. These two functions are obtained using formulas (4) and (5) in section 2.2. The objective and risk-neutral probabilities are computed according to the formulas developed in section 3.2. For simplicity, we will consider one-period options.

5.1 Recursive Utility with State Dependence in Fundamentals and Preferences

We first assume that only the fundamentals are affected by the latent state variables. Therefore, the means and variances of the consumption and dividend growth rates take two values, one in each state, while the three preference parameters β, γ and ρ remain constant. We choose values of the parameters that are close to those estimated in Garcia, Luger, and Renault (2003), where preference parameters are not state-dependent. [10] The legend in Figure 1 reports the values used to compute the risk-neutral and objective probabilities across wealth states and the resulting pricing kernel and absolute risk aversion functions. The left panel contains the marginal risk-neutral and objective distributions. It appears that the risk-neutral distribution has less mass in the center and the right-hand tail, making it more skewed to the left. The center panel in Figure 1 reveals that the unconditional pricing kernel increases in the center return states (over the range of 0.9 to 1.03). We use the term unconditional to emphasize that the pricing kernel function across wealth states is computed using marginal probabilities given by (11) and (15). On the right panel, the corresponding absolute risk aversion function appears to be negative in some regions of the return space.

In Brown and Jackwerth (2000), they propose as an explanation of the puzzles a consumption-based utility model where they introduce as an additional state variable a momentum variable that follows a mean-reverting process. To produce the graphs of the pricing kernels, they simulate 10,000 runs of the processes based on a daily discretization over 31 days and smooth the graph by kernel regression. This corresponds in our setting to the marginal probabilities given by (9) and (13), that is the unconditional average of the probabilities given the trajectory of the state variables. We report the corresponding graphs in Figure 2. In this figure, we also draw the pricing kernel and the absolute risk aversion functions conditional on each of the two states (from the objective probabilities (8) and risk-neutral probabilities (12)). On the left panel, it can be seen that the pricing kernel in each regime declines monotonically across wealth states. However, the unconditional graph exhibits regions where the pricing kernel is locally increasing. In the right panel, we plot the unconditional absolute risk aversion and the regime-dependent functions. For the center wealth states, we find as in Jackwerth (2000) that the risk aversion function becomes negative. Within each regime, the absolute risk aversion functions across wealth states are perfectly decreasing functions of the aggregate wealth: the puzzle disappears.

We next consider state dependence in both the fundamentals and the investor's preference parameters [11]. We investigate several state-dependent preference cases. First, we assume a constant relative risk aversion (CRRA) and a state-dependent elasticity of intertemporal substitution (EIS). Second, we assume a state-dependent risk aversion and a constant EIS. Third, we assume cyclical CRRA and EIS and finally we assume state-dependent time preferences. For all combinations of state-dependent preference parameters, we get very similar results. Therefore, we only report the results for state-dependent relative risk aversion and constant EIS in Figure 3 and Figure 4. For our results, based on state-dependent preferences described below, we disturb preference parameters values around Garcia, Luger, and Renaults (2003) results but still maintain them in a realistic range. Figure 3 reports the results when first the conditioning is done on the past of the state variables and then the marginal probabilities are computed according to equations (11) and (15). In the three panels we observe patterns that are very similar to the respective ones drawn in Figure 1 where only the fundamentals change with the state. The regions over which the pricing kernel increases and the risk aversion becomes negative are displaced to the right and are smaller in magnitude compared to Figure 1. The graphs exhibited in Figure 4 are very similar to the ones we obtained in Figure 2 when only the fundamentals changed.

5.2 External Habit with State Dependence in Beliefs

Campbell and Cochrane (1999) show that an external habit model is able to explain asset pricing puzzles, namely the equity premium puzzle, the risk-free rate puzzle and the volatility puzzle. We show here that an extension of this model proposed by Veronesi (2004) and set in a state-dependent framework can explain the risk aversion puzzle exhibited by Aït-Shalia and Lo (2000) and Jackwerth (2000) based on prices on index options and returns on the index. This extension has the virtue of capturing state-dependent beliefs, since Ziegler (2002) has advocated herogeneity of beliefs as an explanation for the puzzles.

In our calibration we keep the parameters of the fundamentals close to the ones we have used in the examples of the previous section on recursive utility. We have increased slightly the volatility parameters. For the preference parameters, we start from the values used in Veronesi (2004).

Since the fundamentals have different dynamics, we adjust upwards the risk aversion parameter (10 instead of 1.5 to 3). However we reduce the magnitude of the ρ parameter. The parameter ϕ is set at 0.03, close to the value chosen in Veronesi (2004). The resulting risk aversion function across wealth states is drawn in Figure 5. The region over which the function becomes negative is much wider than in the recursive utility case. This suggests that aversion to state uncertainty draws a strong wedge between the risk neutral and the objective probabilities. This should be investigated further in future research.

6. Discussion

As risk-neutral expectations of discounted terminal payoffs, option prices are only informative about the risk-neutral distribution. For example, in Renault (1997), it is shown that, in the presence of a leverage effect, conditioning by the future volatility path has an effect on returns forecasts and implies a very steep and asymmetric volatility smile. However, the volatility smile does not provide information on the potential difference between p and p^*. As we have emphasized in this paper, any difference is well captured by odd shapes of the pricing kernel or the risk aversion function. Therefore, any model that aims at explaining these puzzles must make precise why it captures the relevant differences between p and p^*. Three main approaches can be found in the literature. The difference between the risk-neutral and objective skewness after 1987 has been invoked by Jackwerth (2000) to justify the pricing kernel puzzle. However, one needs to relate the risk-neutral skewness to the objective skewness and explain why they differ. A second line of explanation is based on the fact that p^* is forward looking while p is backward looking. But one must explain why should this imply a difference between the risk-neutral and the objective probabilities in a stationary world. A final potential source of the puzzles has been associated with estimation error, but one may ask why we should observe a persistent discrepancy between p and p^* even when the number of observations is large.

In this paper we provide a unifying framework that gives content to all these explanations. In our setting, there exists hidden state variables which may be unobservable either to the econometrician or the investors or both. In our most general model of section 3., we assume joint log-normality for stock returns and a general SDF conditionally to the path of the state variables. The conditioning set differs whether we consider the future state variables or the past state variables. We have shown that these conditioning issues rationalize the risk aversion puzzles in the sense that, once properly conditioned, the risk aversion functions are consistent with economic theory. Here we will argue that this framework provides a rationale for the three proposed types of explanations.

To link the risk neutral skewness and the objective skewness, we refer to Bakshi, Kapadia and Madan (2003). A way to capture this divergence between the risk-neutral and the physical distributions is through their higher-order moments. Bakshi, Kapadia and Madan show that, in power utility economies (with a pricing kernel exponential in the index returns, $e^{-\gamma R_m}$) , the risk-neutral skewness of the index returns is linked approximately to the higher moments of the physical distribution by the following relation:

$$Skew(t, t+\tau) \approx \overline{Skew}(t, t+\tau) - \gamma(\overline{Kurt}(t, t+\tau) - 3)\overline{Std}(t, t+\tau) \tag{23}$$

where γ denotes the coefficient of relative risk aversion. [12] Therefore, fat tails in the physical distribution will produce a more negatively skewed risk-neutral distribution. Clark (1973) explains that the lack of conditioning implies excess kurtosis. Therefore, the unconditional distribution, which is a mixture of the conditionals upon the state variables distributions, will exhibit fat tails.

According to Ziegler (2002), non-stationarity of the return process may explain that estimates of beliefs obtained from historical frequency return distributions will differ from agents' actual assessments. The presence of the state variables in our setting introduces what Ziegler calls non-stationarity. The observation of the past values of the state variables modifies the expectation about the future values of the state variables and in turn about future returns. At the end, Ziegler (2002) makes the point that the beliefs are heterogeneous and that the researcher ignores beliefs heterogeneity. This is observationally equivalent to our state variable framework. Moreover, we provide a structural model where the beliefs depend on the state variables and which can be estimated once a stochastic process is assumed for the state variables.

Ziegler (2002) also argues that it is hard to rationalize the implied risk aversion smile by beliefs misestimation if agents have homogeneous beliefs. We have seen that it can easily be rationalized by the difference between estimating the unconditional distribution of returns and the conditional one, given state variables. In particular, one cannot exclude the possibility of state dependence in the fundamentals, which will produce similar effects on the estimate of the risk aversion function.

To conclude this discussion, we invoke causality arguments to rationalize the observed puzzles in terms of conditioning. Let $m(U_t^T, \frac{C_T}{C_t})$ the SDF that prices payoffs between t and T. Even though the function $c \to m(u, c)$ is a well decreasing function in c, one may find increasing patterns when simulating paths of (u, c) to obtain a kernel smoother of $E_t\left[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T\right]$. This is what Brown and Jackwerth (2000) and Ziegler (2002) do. The former simulate wealth together with a momentum process, while the latter simulates Pan's (2002) model. It should be emphasized that the puzzle can only occur because C_T is independent from U_t^T, which implies that:

$$E_t\left[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T\right] \neq E_t\left[m(U_t^T, \frac{C_T}{C_t})\right]. \tag{24}$$

Otherwise, a decreasing $c \to m(u,c)$ will imply that $c \to E[m(u,c)]$ will also be decreasing. In our general setting of pricing with state variables, we do obtain that $E_t\left[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T\right]$ is not always decreasing while $c \to m(u,c)$ is always decreasing.

Even though our framework is set with exogenous state variables, it can be shown that the odd patterns in the pricing kernel and the risk aversion functions can be obtained in the model of Heston and Nandi (2000) for more than one-period options. [13]

To rationalize the fact that the pricing kernel puzzle is not maintained when using a proper conditioning, observe that:

$$E_t\left[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T, U_1^t\right] = E_t\left[m(U_t^T, \frac{C_T}{C_t})|U_1^t\right], \tag{25}$$

since C does not cause the state variables U by assumption 2 in a consumption model. Therefore, the decreasing feature of $c \to m(u,c)$ is not modified by the expectation operator.

7. Conclusion

The main goal of this paper was to reconcile with economic theory the puzzling facts about the pricing kernel and the risk aversion functions extracted from option and equity prices. Our central contribution has been to provide a unifying explanation in terms of state dependence when the states may be observed by the investors but not by the researcher. Since the goal of this line of research is to better identify the investors' preferences or beliefs we provide option pricing formulas for several economic models with state dependence. These formulas allow us to recover analytically the risk neutral and objective probabilities across wealth states and thus the risk aversion and pricing kernel functions. We show that models with regime shifts in fundamentals, investor's preferences or beliefs can rationalize the puzzles puzzles put forward in Aït-Sahalia and Lo (2000) and Jackwerth (2000). The absolute risk aversion and pricing kernel functions extracted from the simulated prices in these economies exhibit the same puzzling features as in the original papers and are inconsistent with the usual assumptions of decreasing marginal utility and positive risk aversion. However, the investor utility is well-behaved and her risk aversion remains positive. In other words, investors' behavior is not at odds with economic theory but depends on some factors that the statistician does not observe.

Appendix

Proof of Proposition 3.1. For space considerations, since the proofs are very similar, we refer to the proofs of 4.1, 4.2 and 4.3 for the various steps to be followed to prove the proposition. A fully developed proof is available from the authors upon request. ∎

Proof of Proposition 4.1. Rearranging equation (6.9) for the pricing kernel in Melino and Yang (2003)), one obtain:

$$m_{t+1} = \left[\beta\left(U_t\right)\left(\frac{C_{t+1}}{C_t}\right)^{\rho(U_t)-\frac{\rho(U_t)}{\rho(U_{t+1})}}\right]^{\gamma(U_t)} R_{mt+1}^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-1} P_t^{\frac{\alpha(U_t)}{\rho(U_{t+1})}-\frac{\alpha(U_t)}{\rho(U_t)}},$$

where $\gamma\left(U_t\right) = \frac{\alpha(U_t)}{\rho(U_t)}$ and P_t is the equilibrium price of the market portfolio at time t. If $\rho\left(U_t\right) = \rho\left(U_{t+1}\right)$ and $\beta\left(U_t\right),\ \alpha\left(U_t\right),\ \rho\left(U_t\right)$ are constants, this pricing kernel reduces to the Epstein and Zin (1989) pricing kernel. Let $\varphi\left(U_t\right) = \frac{S_t}{D_t}$ denote the price-dividend ratio and $\lambda\left(U_t\right) = \frac{P_t}{C_t}$ the price-earning ratio. The return on the market portfolio can be written as

$$R_{mt+1} = \frac{P_{t+1}+C_{t+1}}{P_t} = \left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda\left(U_t\right)}\right)\left(\frac{C_{t+1}}{C_t}\right),$$

and the stock return:

$$\frac{S_{t+1}}{S_t} = \frac{\varphi\left(U_1^{t+1}\right)}{\varphi\left(U_t\right)}\frac{D_{t+1}}{D_t}.$$

Let us assume that the conditional probability distribution of $\left(\log\frac{C_{t+1}}{C_t}, \log\frac{D_{t+1}}{D_t}\right)$ given U_1^{t+1} is a bivariate normal:

$$\left[\begin{array}{c}\log\frac{C_{t+1}}{C_t}\\ \log\frac{D_{t+1}}{D_t}\end{array}\right]/U_1^T \rightsquigarrow N\left[\left(\begin{array}{c}\mu_{X_{t+1}}\\ \mu_{Y_{t+1}}\end{array}\right),\left(\begin{array}{cc}\sigma_{X_{t+1}}^2 & \sigma_{XY,t+1}\\ \sigma_{XY,t+1} & \sigma_{Y_{t+1}}^2\end{array}\right)\right], \tag{A.1}$$

with $U_1^{t+1} = \left(U_\tau\right)_{1\leq\tau\leq t+1}$. Taking the logarithm of m_{t+1}, we get

$$\begin{aligned}\log m_{t+1} = {} & \gamma\left(U_t\right)\log\beta\left(U_t\right) + \left(\frac{\alpha\left(U_t\right)}{\rho\left(U_{t+1}\right)}-1\right)\log\left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda\left(U_t\right)}\right) + \left(\frac{\alpha\left(U_t\right)}{\rho\left(U_{t+1}\right)}-\frac{\alpha\left(U_t\right)}{\rho\left(U_t\right)}\right)\log\left(\lambda\left(U_t\right)C_t\right) + \\ & \left[\gamma\left(U_t\right)\left(\rho\left(U_t\right)-\frac{\rho\left(U_t\right)}{\rho\left(U_{t+1}\right)}\right)+\left(\frac{\alpha\left(U_t\right)}{\rho\left(U_{t+1}\right)}-1\right)\right]\log\left(\frac{C_{t+1}}{C_t}\right).\end{aligned}$$

The logarithm of the stock return is

$$\log\frac{S_{t+1}}{S_t} = \log\frac{\varphi\left(U_1^{t+1}\right)}{\varphi\left(U_t\right)} + \log\frac{D_{t+1}}{D_t}.$$

Consequently,

$$\left[\begin{array}{c}\log m_{t+1}\\ \log \frac{S_{t+1}}{S_t}\end{array}\right] = A + B\left[\begin{array}{c}\log \frac{C_{t+1}}{C_t}\\ \log \frac{D_{t+1}}{D_t}\end{array}\right],$$

where $A = (a_1, a_2)'$ with

$$\begin{aligned}
a_1 &= \gamma(U_t)\log\beta(U_t) + \left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right)\log\left(\frac{\lambda\left(U_1^{t+1}\right)+1}{\lambda(U_t)}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - \frac{\alpha(U_t)}{\rho(U_t)}\right)\log(\lambda(U_t)C_t),\\
a_2 &= \log\frac{\varphi\left(U_1^{t+1}\right)}{\varphi(U_t)},
\end{aligned}$$

and B is a diagonal matrix with diagonal coefficients:

$$\begin{aligned}
b_{11} &= \left[\gamma(U_t)\left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1\right)\right],\\
b_{22} &= 1.
\end{aligned}$$

Using (A.1), it is straightforward to show:

$$\left[\begin{array}{c}\log m_{t+1}\\ \log \frac{S_{t+1}}{S_t}\end{array}\right]/U_1^{t+1} \rightsquigarrow N[\mu, \Sigma_{ms}]$$

with

$$\begin{aligned}
\mu &= A + B\begin{pmatrix}\mu_{X_{t+1}}\\ \mu_{Y_{t+1}}\end{pmatrix},\\
\Sigma_{ms} &= B\begin{pmatrix}\sigma^2_{X_{t+1}} & \sigma_{X_{t+1}Y_{t+1}}\\ \sigma_{X_{t+1}Y_{t+1}} & \sigma^2_{Y_{t+1}}\end{pmatrix}B.
\end{aligned}$$

This completes the proof. ∎

Proof of Proposition 4.2. To demonstrate proposition 3.3, we first consider the following lemma:

Lemma: If the random variable (Z_1, Z_2) is normally distributed:

$$\left[\begin{array}{c}Z_1\\ Z_2\end{array}\right] \rightsquigarrow N\left(\begin{pmatrix}\mu_{Z_1}\\ \mu_{Z_2}\end{pmatrix}, \left[\begin{array}{cc}\sigma^2_{Z_1} & \rho\sigma_{Z_1}\sigma_{Z_2}\\ \rho\sigma_{Z_1}\sigma_{Z_2} & \sigma^2_{Z_2}\end{array}\right]\right).$$

Let denote Q, the probability measure corresponding to the process (Z_1, Z_2) and define Q^*, the probability measure defined by:

$$\frac{dQ^*}{dQ}(Z) = \exp\left[(Z - \mu_{Z_1}) - \frac{\sigma^2_{Z_1}}{2}\right].$$

Then, by Girsanov theorem,

$$Q^*\left(Z_2 \geq 0\right) = 1 - \Phi\left[\frac{-\mu_{Z_2} - \rho\sigma_{Z_1}\sigma_{Z_2}}{\sigma_{Z_2}}\right] = \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_1}\right] \tag{A.2}$$

and

$$\begin{aligned} E\left[\exp\left(Z_1\right) 1_{Z_2 \geq 0}\right] &= \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right] \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_1}\right] \\ &= \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right] Q^*\left(Z_2 \geq 0\right), \end{aligned}$$

where Φ is the cumulative normal distribution function.

At time t, a European call option on the dividend-paying stock with strike K that expires at T is worth

$$\pi_t = E_t\left[m_t^T \max\left(S_T + D_T - K, 0\right)\right],$$

with

$$m_t^T = \prod_{\tau=t}^{T-1} m_{\tau+1}.$$

Let denote U_1^T the trajectory of state variables form t up to T. So that

$$\begin{aligned} \frac{\pi_t}{S_t} &= E_t\left[m_t^T \max\left(\frac{S_T + D_T}{S_t} - \frac{K}{S_t}, 0\right)\right] \\ &= E_t E_t\left[m_t^T \max\left(\frac{S_T + D_T}{S_t} - \frac{K}{S_t}, 0\right) | U_1^T\right] \\ &= E_t\left[H\left(U_t^T\right) - \frac{K}{S_t} G\left(U_t^T\right)\right] \end{aligned}$$

With:

$$H\left(U_t^T\right) = E_t\left[m_t^T\left(\frac{S_T + D_T}{S_t}\right) 1_{\frac{S_T + D_T}{S_t} > \frac{K}{S_t}} | U_1^T\right] \text{ and } G\left(U_t^T\right) = E_t\left[m_t^T 1_{\frac{S_T + D_T}{S_t} > \frac{K}{S_t}} | U_1^T\right].$$

We denote:

$$Z_1 = \log m_t^T = \sum_{\tau=t}^{T-1} \log m_{\tau+1} \text{ and } Z_2 = \log\frac{S_T + D_T}{S_t} - \log\frac{K}{S_t}.$$

We first find the distribution of Z_1 and Z_2. To derive derive Z_1, we use the pricing kernel of the form:

$$m_{\tau+1} = \left[\beta\left(U_\tau\right)\left(\frac{C_{\tau+1}}{C_\tau}\right)^{\rho(U_\tau) - \frac{\rho(U_\tau)}{\rho(U_{\tau+1})}}\right]^{\gamma(U_\tau)} R_{m\tau+1}^{\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - 1} P_\tau^{\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau)}.$$

see equation 14 in our paper. Taking the log of this expression gives:

$$\begin{aligned} \log m_{\tau+1} = {} & \gamma\left(U_\tau\right)\log\left(\beta\left(U_\tau\right)\right) + \gamma\left(U_\tau\right)\left(\rho\left(U_\tau\right) - \frac{\rho\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)}\right)\log\left(\frac{C_{\tau+1}}{C_\tau}\right) + \\ & \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\log R_{m\tau+1} + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(P_\tau\right). \end{aligned}$$

Note that the equilibrium model characterizes the asset prices S_τ and P_τ as:

$$\begin{aligned}\frac{S_\tau}{D_\tau} &= \varphi\left(U_1^\tau\right),\\ \frac{P_\tau}{C_\tau} &= \lambda\left(U_1^\tau\right).\end{aligned}$$

Thus, the market return is:

$$R_{m\tau+1} = \frac{P_{\tau+1}+C_{\tau+1}}{P_\tau} = \frac{\lambda\left(U_1^{\tau+1}\right)+1}{\lambda\left(U_1^\tau\right)}\frac{C_{\tau+1}}{C_\tau}.$$

We now expand the log-pricing kernel, $\log m_{\tau+1}$, as follows:

$$\begin{aligned}\log m_{\tau+1} &= \gamma\left(U_\tau\right)\log\left(\beta\left(U_\tau\right)\right) + \left[\gamma\left(U_\tau\right)\left(\rho\left(U_\tau\right) - \frac{\rho\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)}\right) + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\right]\log\left(\frac{C_{\tau+1}}{C_\tau}\right) + \\ &\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\log\left(\frac{\lambda\left(U_1^{\tau+1}\right)+1}{\lambda\left(U_1^\tau\right)}\right) + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(\lambda\left(U_1^\tau\right)\right) + \\ &+ \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(C_\tau\right)\end{aligned}$$

Consequently the random variable Z_1 is:

$$\begin{aligned}Z_1 &= \sum_{\tau=t}^{T-1}\gamma\left(U_\tau\right)\log\left(\beta\left(U_\tau\right)\right) + \\ &\sum_{\tau=t}^{T-1}\left[\gamma\left(U_\tau\right)\left(\rho\left(U_\tau\right) - \frac{\rho\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)}\right) + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\right]\log\left(\frac{C_{\tau+1}}{C_\tau}\right) + \\ &\sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\log\left(\frac{\lambda\left(U_1^{\tau+1}\right)+1}{\lambda\left(U_1^\tau\right)}\right) + \sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(\lambda\left(U_1^\tau\right)\right) + \\ &+ \sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(C_\tau\right)\end{aligned} \tag{A.3}$$

Note that

$$C_\tau = C_t\prod_{\iota=t}^{\tau-1}\frac{C_{\iota+1}}{C_\iota} \text{ for } \tau > t.$$

We replace C_τ in Z_1 and deduce:

$$\begin{aligned}Z_1 &= \psi_1 + \sum_{\tau=t}^{T-1}\left[\gamma\left(U_\tau\right)\left(\rho\left(U_\tau\right) - \frac{\rho\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)}\right) + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\right]\log\left(\frac{C_{\tau+1}}{C_\tau}\right) + \\ &\sum_{\tau=t+1}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\sum_{\iota=t}^{\tau-1}\log\frac{C_{\iota+1}}{C_\iota},\end{aligned} \tag{A.4}$$

where ψ_1 is given as follows:

$$\begin{aligned}\psi_1 &= \sum_{\tau=t}^{T-1}\gamma\left(U_\tau\right)\log\left(\beta\left(U_\tau\right)\right) + \sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(C_t\right) + \\ &\sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right)\log\left(\frac{\lambda\left(U_1^{\tau+1}\right)+1}{\lambda\left(U_1^\tau\right)}\right) + \sum_{\tau=t}^{T-1}\left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right)\right)\log\left(\lambda\left(U_1^\tau\right)\right),\end{aligned}$$

We now compute Z_2 :

$$Z_2 = \log \frac{S_T + D_T}{S_t} - \log \frac{K}{S_t} = \psi_2 + \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau},$$

with

$$\psi_2 = \sum_{\tau=t}^{T-1} \log \frac{\left(1 + \varphi\left(U_1^{\tau+1}\right)\right)}{\varphi\left(U_1^{\tau}\right)} - \log \frac{K}{S_t}.$$

Under Assumption 3', given the state variable trajectory, the random variable

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \sum_{\tau=t}^{T-1} \begin{bmatrix} a_{1\tau+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \log \frac{C_{\tau+1}}{C_\tau} \\ \log \frac{D_{\tau+1}}{D_\tau} \end{bmatrix} + \sum_{\tau=t+1}^{T-1} \begin{bmatrix} a_{0\tau+1} \\ 0 \end{bmatrix} \sum_{\iota=t}^{\tau-1} \log \frac{C_{\iota+1}}{C_\iota},$$

with

$$\begin{aligned} a_{0\tau+1} &= \frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - \gamma\left(U_\tau\right), \\ a_{1\tau+1} &= \gamma\left(U_\tau\right)\left(\rho\left(U_\tau\right) - \frac{\rho\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)}\right) + \left(\frac{\alpha\left(U_\tau\right)}{\rho\left(U_{\tau+1}\right)} - 1\right) \end{aligned}$$

is normally distributed with conditional mean vector characterized by:

$$\begin{aligned} \mu_{Z_1} &= \psi_1 + \sum_{\tau=t}^{T-1} a_{1\tau+1}\mu_{X\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1} \mu_{X\iota+1}\right), \\ \mu_{Z_2} &= \psi_2 + \sum_{\tau=t}^{T-1} \mu_{Y\tau+1}, \end{aligned}$$

and conditional variance covariance matrix defined by

$$\begin{aligned} \sigma_{Z_1}^2 &= \sum_{\tau=t}^{T-1} a_{1\tau+1}^2 \sigma_{X\tau+1}^2 + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}^2 \sum_{\iota=t}^{\tau-1} \sigma_{X\iota+1}^2 + \\ &\quad 2 \sum_{t \le i < j \le T-1} a_{0i+1}a_{0j+1} Cov\left(\sum_{\iota=t}^{i-1} \log \frac{C_{\iota+1}}{C_\iota}, \sum_{\iota'=t}^{j-1} \log \frac{C_{\iota'+1}}{C_{\iota'}}\right) \\ &= \sum_{\tau=t}^{T-1} a_{1\tau+1}^2 \sigma_{X\tau+1}^2 + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}^2 \sum_{\iota=t}^{\tau-1} \sigma_{X\iota+1}^2 + 2 \sum_{t \le i < j \le T-1} a_{0i+1}a_{0j+1} \sum_{\iota=t}^{i-1} \sigma_{X\iota+1}^2 \\ \sigma_{Z_2}^2 &= \sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2 \\ \rho\sigma_{Z_1}\sigma_{Z_2} &= Cov\left(\sum_{\tau=t}^{T-1} a_{1\tau+1} \log\left(\frac{C_{\tau+1}}{C_\tau}\right), \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau}\right) + Cov\left(\sum_{\tau=t+1}^{T-1} a_{0\tau+1} \sum_{\iota=t}^{\tau-1} \log \frac{C_{\iota+1}}{C_\iota}, \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau}\right) \\ &= \sum_{\tau=t}^{T-1} a_{1\tau+1}\sigma_{XY,\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1} \sigma_{XY,\iota+1}\right) \end{aligned}$$

Second, we use the lemma above (see the proof of proposition 4.2) to deduce:

$$\begin{aligned} G\left(U_t^T\right) &= E_t\left[m_t^T 1_{\log \frac{S_T+D_T}{S_t}-\log \frac{K}{S_t}>0}|U_1^T\right] \\ &= \exp\left[\mu_{Z_1}+\frac{\sigma_{Z_1}^2}{2}\right] Q^*\left(Z_2\geq 0\right) \\ &= \exp\left[\mu_{Z_1}+\frac{\sigma_{Z_1}^2}{2}\right] \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}}+\rho\sigma_{Z_1}\right] \end{aligned}$$

Let denote $B\left(t,T\right)$ the bond price:

$$B\left(t,T\right)=E_t\widetilde{B}\left(t,T\right),$$

with $\widetilde{B}\left(t,T\right)=E_t\left[m_t^T|U_1^T\right]$. We realize that

$$\widetilde{B}\left(t,T\right)=\exp\left[\mu_{Z_1}+\frac{\sigma_{Z_1}^2}{2}\right].$$

Denote $d_2\left(x_t\right)=\frac{\mu_{Z_2}}{\sigma_{Z_2}}+\rho\sigma_{Z_1}$. We have

$$d_2\left(x_t\right)==\frac{\sum_{\tau=t}^{T-1}\log\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}-\log\frac{K}{S_t}+\sum_{\tau=t}^{T-1}\mu_{Y\tau+1}+\sum_{\tau=t}^{T-1}a_{1\tau+1}\sigma_{XY,\tau+1}+\sum_{\tau=t+1}^{T-1}a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\sigma_{XY,\iota+1}\right)}{\sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}} \tag{A.5}$$

To simplify this last expression, consider the Euler equation:

$$E_t\left[m_t^T\frac{\left(S_T+D_T\right)}{S_t}\right]=1. \tag{A.6}$$

Expression (A.6) can be rewritten as

$$E_t\left(Q_{XY}\left(t,T\right)\right)=1,$$

where:

$$\begin{aligned} Q_{XY}\left(t,T\right) &= E_t\left[m_t^T\frac{\left(S_T+D_T\right)}{S_t}|U_1^T\right] \\ &= \widetilde{B}\left(t,T\right)E_t\left[\frac{\left(S_T+D_T\right)}{S_t}|U_1^T\right]\exp\left[\sum_{\tau=t}^{T-1}a_{1\tau+1}\sigma_{XY,\tau+1}+\sum_{\tau=t+1}^{T-1}a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\sigma_{XY,\iota+1}\right)\right] \end{aligned}$$

with

$$E_t\left[\frac{S_T+D_T}{S_t}|U_1^T\right]=\exp\left[\mu_{Z_2}+\log\frac{K}{S_t}+\frac{1}{2}\sigma_{Z_2}^2\right].$$

We now simplify (A.5) as

$$d_2(x_t) = \frac{x_t + \log\left(Q_{XY}(t,T)\,\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}{\sqrt{\sum_{\tau=t}^{T-1}\sigma^2_{Y\tau+1}}},$$

with

$$x_t = \log\frac{S_t}{KB(t,T)}.$$

As a result,

$$G\left(U_t^T\right) = \widetilde{B}(t,T)\,\Phi\left(d_2(x_t)\right).$$

Third, we compute $H\left(U_t^T\right)$:

$$\begin{aligned} H\left(U_t^T\right) &= E_t\left[m_t^T\left(\frac{S_T+D_T}{S_t}\right)1_{\frac{S_T+D_T}{S_t}>\frac{K}{S_t}}|U_1^T\right] \\ &= E_t\left[\exp\left(\sum_{\tau=t}^{T-1}\log m_{\tau+1} + \sum_{\tau=t}^{T-1}\log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}\right)\frac{D_{\tau+1}}{D_\tau}\right)\right)1_{\log\frac{S_T+D_T}{S_t}>\log\frac{K}{S_t}}|U_1^T\right]. \end{aligned}$$

Let denote

$$Z_3 = \sum_{\tau=t}^{T-1}\log m_{\tau+1} + \sum_{\tau=t}^{T-1}\log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}\right)\frac{D_{\tau+1}}{D_\tau}\right).$$

We use the lemma above (see the proof of proposition 4.2) to obtain:

$$\begin{aligned} H\left(U_t^T\right) &= \exp\left[\mu_{Z_3} + \frac{\sigma^2_{Z_3}}{2}\right]Q^*\left(Z_2 \geq 0\right) \\ &= \exp\left[\mu_{Z_3} + \frac{\sigma^2_{Z_3}}{2}\right]\Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_3}\right]. \end{aligned}$$

But,

$$Q_{XY}(t,T) = E_t\left[m_t^T\frac{(S_T+D_T)}{S_t}|U_1^T\right] = E_t\left[\exp Z_3|U_1^T\right].$$

Under assumption A3' the random variable Z_3 is normally distributed. Consequently,

$$Q_{XY}(t,T) = \exp\left[\mu_{Z_3} + \frac{\sigma^2_{Z_3}}{2}\right].$$

We denote

$$d_1(x_t) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_3} = \frac{\mu_{Z_2+}\rho\sigma_{Z_3}\sigma_{Z_2}}{\sigma_{Z_2}}.$$

But,

$$\begin{aligned}\rho\sigma_{Z_2}\sigma_{Z_3} &= \rho\sigma_{Z_1}\sigma_{Z_2} + Var\left(\sum_{\tau=t}^{T-1}\log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}\right)\frac{D_{\tau+1}}{D_\tau}\right)|U_1^T\right)\\ &= \rho\sigma_{Z_1}\sigma_{Z_2} + \sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2.\end{aligned}$$

As a result

$$d_1(x_t) = \frac{\mu_{Z_2}+\rho\sigma_{Z_1}\sigma_{Z_2}+\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}{\sigma_{Z_2}} = d_2(x_t) + \sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}$$

and:

$$H\left(U_t^T\right) = Q_{XY}(t,T)\,\Phi\left[d_1(x_t)\right].$$

Finally the call option price is given by:

$$\frac{\pi_t}{S_t} = E_t\left[Q_{XY}(t,T)\,\Phi\left[d_1(x_t)\right] - \frac{K}{S_t}\widetilde{B}(t,T)\,\Phi\left(d_2(x_t)\right)\right].$$

∎

Proof of Proposition 4.3. At time t, a European call option on the dividend-paying stock with strike K is worth:

$$\begin{aligned}\frac{\pi_t}{S_t} &= E_t\left[m_t^T\max\left(\frac{S_T+D_T}{S_t}-\frac{K}{S_t},0\right)\right]\\ &= E_tE_t\left[m_t^T\max\left(\frac{S_T+D_T}{S_t}-\frac{K}{S_t},0\right)|U_1^T\right]\\ &= E_t\left[H\left(U_t^T\right)-\frac{K}{S_t}G\left(U_t^T\right)\right]\end{aligned}$$

With:

$$H\left(U_t^T\right) = E_t\left[m_t^T\left(\frac{S_T+D_T}{S_t}\right)1_{\frac{S_T+D_T}{S_t}>\frac{K}{S_t}}|U_1^T\right] \text{ and } G\left(U_t^T\right) = E_t\left[m_t^T 1_{\frac{S_T+D_T}{S_t}>\frac{K}{S_t}}|U_1^T\right].$$

We denote:

$$Z_1 = \log m_t^T = \sum_{\tau=t}^{T-1}\log m_{\tau+1} \text{ and } Z_2 = \log\frac{S_T+D_T}{S_t} - \log\frac{K}{S_t}.$$

Then,

$$\begin{aligned}Z_1 &= -\phi(T-t) - (\rho(U_{T-1})+\alpha)\log\left(\frac{C_T}{C_{T-1}}\right) + \sum_{\tau=t}^{T-2}\left[\rho(U_{\tau+1}) - (\rho(U_\tau)+\alpha)\right]\log\left(\frac{C_{\tau+1}}{C_\tau}\right) +\\ &\quad \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right)\\ Z_2 &= \sum_{\tau=t}^{T-1}\log\left(\frac{1+\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^{\tau}\right)}\right) + \sum_{\tau=t}^{T-1}\log\left(\frac{D_{\tau+1}}{D_\tau}\right) - \log\frac{K}{S_t}\end{aligned}$$

Denote $B(t,T)$ the bond price:

$$B(t,T) = E_t \widetilde{B}(t,T),$$

with $\widetilde{B}(t,T) = E_t\left[m_t^T | U_1^T\right]$. Under Assumptions 1, 2 and 3', we have,

$$\widetilde{B}(t,T) = \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right].$$

where

$$\begin{aligned}
\mu_{Z_1} &= -\phi(T-t) - (\rho(U_{T-1}) + \alpha)\mu_{X_T} \\
&\quad + \sum_{\tau=t}^{T-2} [\rho(U_{\tau+1}) - (\rho(U_\tau) + \alpha)]\mu_{X_{\tau+1}} + \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right) \\
\sigma_{Z_1}^2 &= (\rho(U_{T-1}) + \alpha)^2 \sigma_{X_T}^2 + \sum_{\tau=t}^{T-2} [\rho(U_{\tau+1}) - (\rho(U_\tau) + \alpha)]^2 \sigma_{X_{\tau+1}}^2
\end{aligned}$$

and

$$\begin{aligned}
\mu_{Z_2} &= \sum_{\tau=t}^{T-1} \log\left(\frac{1+\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^\tau\right)}\right) + \sum_{\tau=t}^{T-1} \mu_{Y_{\tau+1}} - \log\frac{K}{S_t} \\
\sigma_{Z_2}^2 &= \sum_{\tau=t}^{T-1} \sigma_{Y_{\tau+1}}^2
\end{aligned}$$

We now consider the Euler equation:

$$E_t\left[m_t^T \frac{(S_T + D_T)}{S_t}\right] = 1.$$

which can be expressed as:

$$E_t(Q_{XY}(t,T)) = 1,$$

with

$$Q_{XY}(t,T) = E_t\left[m_t^T \frac{S_T + D_T}{S_t} | U_1^T\right]$$

Let denote

$$Z_3 = \sum_{\tau=t}^{T-1} \log m_{\tau+1} + \sum_{\tau=t}^{T-1} \log\left(\frac{1+\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^\tau\right)} \frac{D_{\tau+1}}{D_\tau}\right).$$

Therefore, under assumptions 1, 2 and 3',

$$\begin{aligned}
Q_{XY}(t,T) &= E_t\left[m_t^T \frac{(S_T + D_T)}{S_t} | U_1^T\right] \\
&= \widetilde{B}(t,T)\, E\left(\frac{(S_T + D_T)}{S_t} | U_1^T\right) \exp(\psi)
\end{aligned}$$

where

$$\psi = \sum_{\tau=t}^{T-2} \left(\rho\left(U_{\tau+1}\right) - \left(\rho\left(U_{\tau}\right) + \alpha\right)\right) \sigma_{XY,\tau+1} - \left(\rho\left(U_{T-1}\right) + \alpha\right) \sigma_{XY,T}$$

$$E\left(\frac{\left(S_T + D_T\right)}{S_t} | U_1^T\right) = \exp\left(\sum_{\tau=t}^{T-1} \log\left(\frac{1+\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^{\tau}\right)}\right) + \sum_{\tau=t}^{T-1} \mu_{Y\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma^2_{Y\tau+1}\right)$$

We use the lemma above (see the proof of proposition 4.2) to compute $G\left(U_t^T\right)$

$$\begin{aligned} G\left(U_t^T\right) &= E_t\left[m_t^T 1_{\log \frac{S_T+D_T}{S_t} - \log \frac{K}{S_t} > 0} | U_1^T\right] \\ &= \exp\left[\mu_{Z_1} + \frac{\sigma^2_{Z_1}}{2}\right] Q^*\left(Z_2 \geq 0\right) \\ &= \exp\left[\mu_{Z_1} + \frac{\sigma^2_{Z_1}}{2}\right] \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_1}\right] \end{aligned}$$

Denote $d_2\left(x_t\right) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_1}$. We have:

$$\begin{aligned} d_2\left(x_t\right) &= \frac{\sum_{\tau=t}^{T-1} \log\left(\frac{1+\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^{\tau}\right)}\right) + \sum_{\tau=t}^{T-1} \mu_{Y_{\tau+1}} - \log\frac{K}{S_t} + \psi}{\sqrt{\sum_{\tau=t}^{T-1} \sigma^2_{Y\tau+1}}} \\ &= \frac{x_t + \log\left(Q_{XY}\left(t,T\right)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma^2_{Y\tau+1}}{\sqrt{\sum_{\tau=t}^{T-1} \sigma^2_{Y\tau+1}}} \end{aligned}$$

with

$$x_t = \log\frac{S_t}{KB\left(t,T\right)}.$$

As a result,

$$G\left(U_t^T\right) = \widetilde{B}\left(t,T\right)\Phi\left(d_2\left(x_t\right)\right).$$

Now, we compute $H\left(U_t^T\right)$:

$$\begin{aligned} H\left(U_t^T\right) &= E_t\left[m_t^T\left(\frac{S_T+D_T}{S_t}\right) 1_{\frac{S_T+D_T}{S_t} > \frac{K}{S_t}} | U_1^T\right] \\ &= E_t\left[\exp\left(\sum_{\tau=t}^{T-1} \log m_{\tau+1} + \sum_{\tau=t}^{T-1} \log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}\right)\frac{D_{\tau+1}}{D_\tau}\right)\right) 1_{\log \frac{S_T+D_T}{S_t} > \log\frac{K}{S_t}} | U_1^T\right]. \end{aligned}$$

Let denote

$$Z_3 = \sum_{\tau=t}^{T-1} \log m_{\tau+1} + \sum_{\tau=t}^{T-1} \log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right)+1\right)}{\varphi\left(U_1^{\tau}\right)}\right)\frac{D_{\tau+1}}{D_\tau}\right).$$

We use the lemma above (see the proof of proposition 4.2) to compute $H\left(U_t^T\right)$:

$$\begin{aligned} H\left(U_t^T\right) &= \exp\left[\mu_{Z_3} + \frac{\sigma_{Z_3}^2}{2}\right] Q^*\left(Z_2 \geq 0\right) \\ &= \exp\left[\mu_{Z_3} + \frac{\sigma_{Z_3}^2}{2}\right] \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_3}\right]. \end{aligned}$$

But,

$$Q_{XY}\left(t,T\right) = E_t\left[m_t^T \frac{\left(S_T + D_T\right)}{S_t} | U_1^T\right] = E_t\left[\exp Z_3 | U_1^T\right].$$

Under assumption A3', the random variable Z_3 is normally distributed. Thus:

$$Q_{XY}\left(t,T\right) = \exp\left[\mu_{Z_3} + \frac{\sigma_{Z_3}^2}{2}\right].$$

We denote

$$d_1\left(x_t\right) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho\sigma_{Z_3} = \frac{\mu_{Z_2 + }\rho\sigma_{Z_3}\sigma_{Z_2}}{\sigma_{Z_2}}.$$

But,

$$\begin{aligned} \rho\sigma_{Z_2}\sigma_{Z_3} &= \rho\sigma_{Z_1}\sigma_{Z_2} + Var\left(\sum_{\tau=t}^{T-1} \log\left(\left(\frac{\left(\varphi\left(U_1^{\tau+1}\right) + 1\right)}{\varphi\left(U_1^{\tau}\right)}\right) \frac{D_{\tau+1}}{D_\tau}\right) | U_1^T\right) \\ &= \rho\sigma_{Z_1}\sigma_{Z_2} + \sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2. \end{aligned}$$

As a result

$$d_1\left(x_t\right) = \frac{\mu_{Z_2 + }\rho\sigma_{Z_1}\sigma_{Z_2} + \sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2}{\sigma_{Z_2}} = d_2\left(x_t\right) + \sqrt{\sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2}$$

and:

$$H\left(U_t^T\right) = Q_{XY}\left(t,T\right) \Phi\left[d_1\left(x_t\right)\right].$$

Finally the call option price is given by:

$$\frac{\pi_t}{S_t} = E_t\left[Q_{XY}\left(t,T\right) \Phi\left[d_1\left(x_t\right)\right] - \frac{K}{S_t} \widetilde{B}\left(t,T\right) \Phi\left(d_2\left(x_t\right)\right)\right].$$

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Notes

[1]The authors also estimate an orthogonal polynomial pricing kernel and find that it exhibits some of the risk-aversion characteristics noted by Jackwerth (2000), with a region of negative absolute risk aversion over the range from 4% to 2% for returns and an increasing absolute risk aversion for returns greater than 4%.

[2]The authors also estimate a CCRA expected utility model and find a similar variability of the estimates found as in the related studies. The average value is 7.2 over the 1991-1995 period with a standard deviation of 4.83.

[3] Liu et al. (2004) propose a parametric closed-form transformation from risk-neutral to real-world distributions based on a mixture of two lognormal densities but without any particular economic interpretation.

[4]We refer to the date of the first issue of this working paper, but some material is found only in the updated version of 2004.

[5] David and Veronesi (1999) also propose an incomplete information model where investors' uncertainty about the dividend process explains the intertemporal variation in the slope and curvature of implied volatility curves.

[6] They notice that "as few as 8 option prices seem to contain enough information to determine the general shape of the implied distribution" and that "at the extreme, the constraints themselves will completely determine the solution".

[7] This model extends the bond pricing model of Constantinides (1992). In the latter, Assumption 1 is only maintained for the SDF sequence (m_t). Resulting bond prices were therefore deterministic functions of the state variables, and Assumption 1 becomes trivial with S_t viewed as a bond price. A second extension relates to the processes considered for the state variables. While Constantinides (1992) considers only AR(1) processes, our setting accommodates any process. In particular, we have in mind Markov switching regime models which can capture any kind of stochastic volatility and jumps in the return process as well as in the volatility.

[8]It is a conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\overline{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\overline{\sigma}_{t,T}$. It nests three well-known models: the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic; the Hull and White (1987)stochastic volatility extension, since $\overline{\sigma}^2_{t,T} = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the integrated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting; the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992).

[9]See also Gordon and St Amour (2000) for an alternative way to introduce state dependence in preferences in a CCAPM framework.

[10]Garcia, Luger, and Renault (2003) estimate a utility-based option-pricing model in the recursive utility framework of Epstein and Zin (1989). They use daily price data for S&P 500 index European call options obtained from the

Chicago Board Options Exchange for the period January 1991 to December 1995. They also use daily return data for the S&P 500 index and estimate preferences parameters in S&P 500 options prices. They find quite reasonable values for the coefficient of the risk aversion and the intertemporal elasticity of substitution.

[11]For space considerations, we do not report the case where the fundamentals remain constant and the preference parameters change with the state. The results obtained were very similar to the other reported cases.

[12] Bakshi, Kapadia and Madan show that this relation can be generalized to a broader family of utility functions. In particular, in the case of state-dependent utilities, the skew dynamics can depend on conditional moves in risk aversion.

[13]We derive the expressions for the risk-neutral and objective distributions and produce graphs similar to the ones in section 5..They are available upon request from the authors.

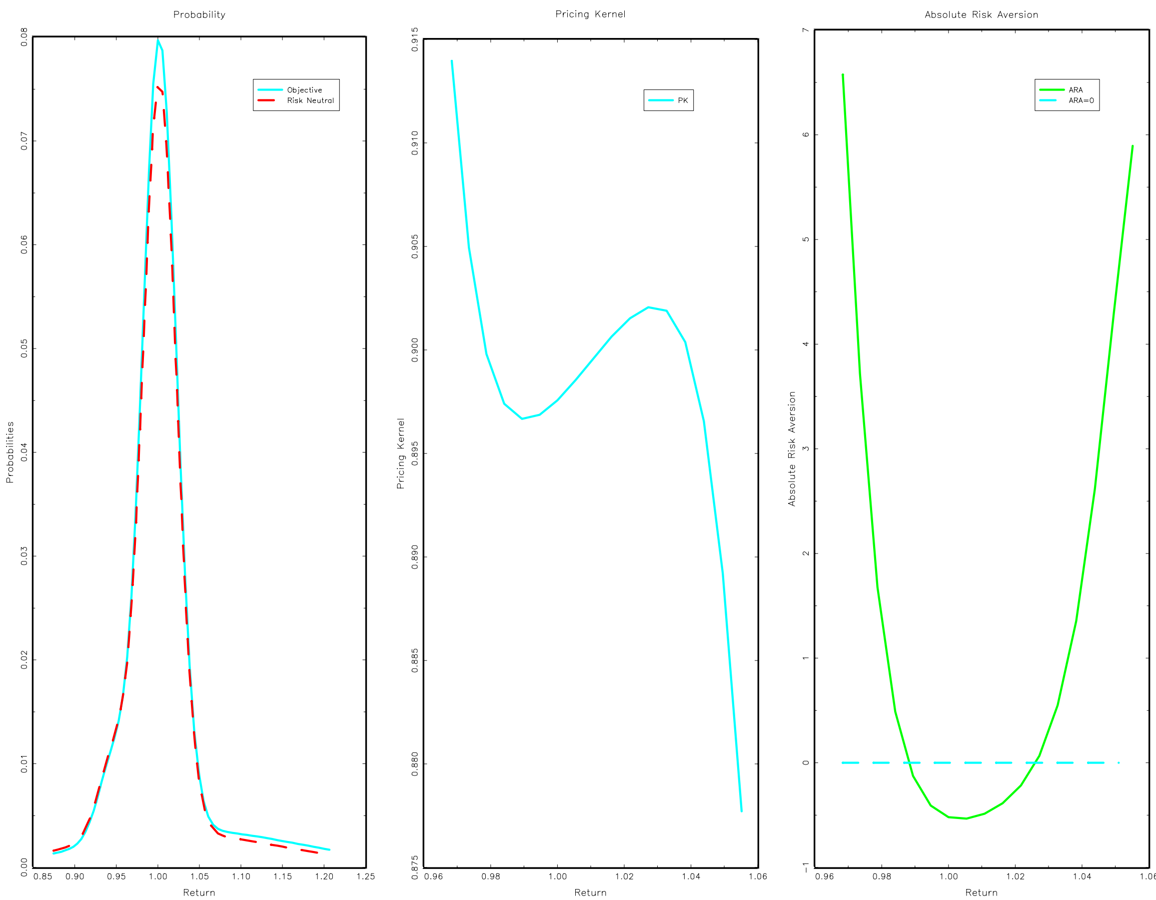

Figure 1: **Risk-Neutral and Objective Distributions, Pricing Kernel (PK) and Absolute Risk Aversion (ARA) functions with state dependence in fundamentals**. The preference parameters are: $\beta = 0.95$, $\alpha = -5$, $\rho = -11$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividends is 0.6. The number of wealth states is $n = 170$. The left-hand panel contains the marginal objective and risk-neutral distributions based on equations (11) and (15). The middle contains the corresponding pricing kernel (PK) function. The right-hand panel contains the absolute risk aversion (ARA) function across wealth states.

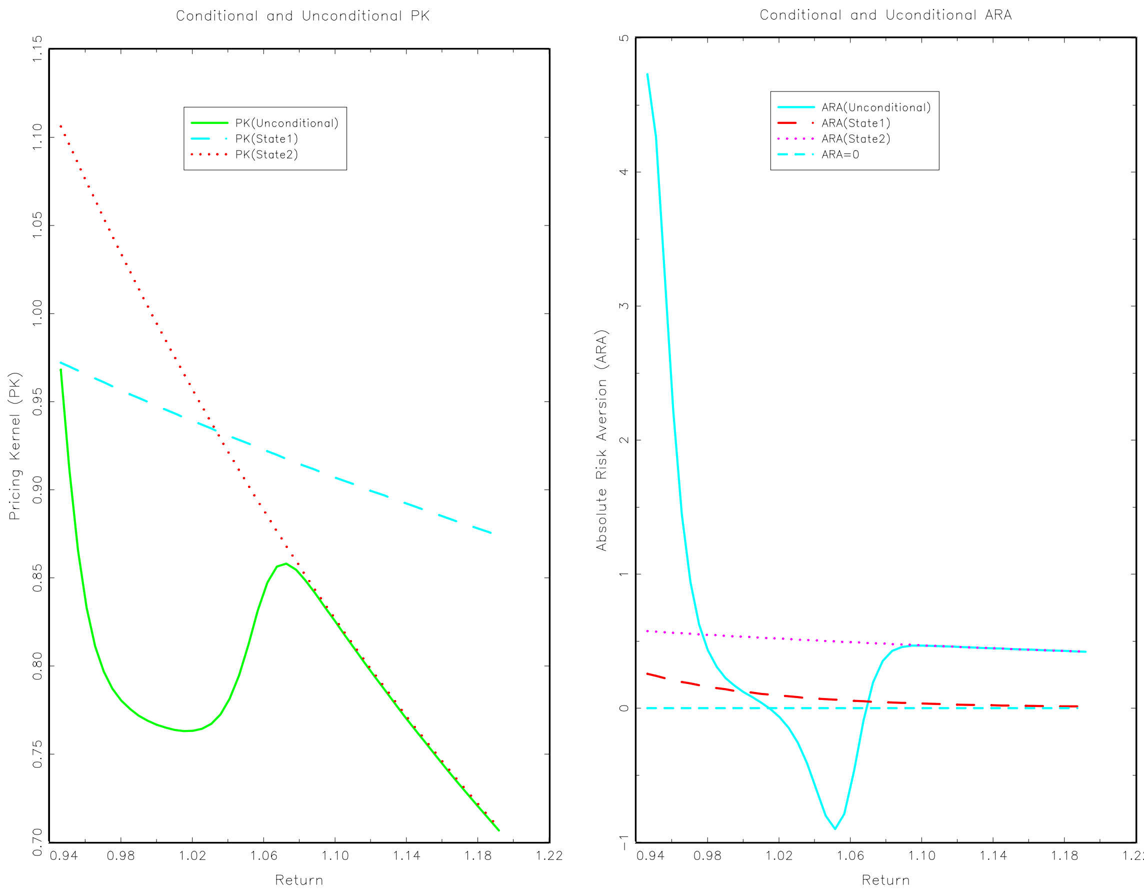

Figure 2: **Pricing Kernel (PK) and Absolute Risk Aversion (ARA) functions with state dependence in fundamentals.** The preference parameters are: $\beta = 0.95$, $\alpha = -5$, $\rho = -11$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividends is 0.6. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional PK functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in equations (8) and (12). The unconditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in equations (9) and (13).

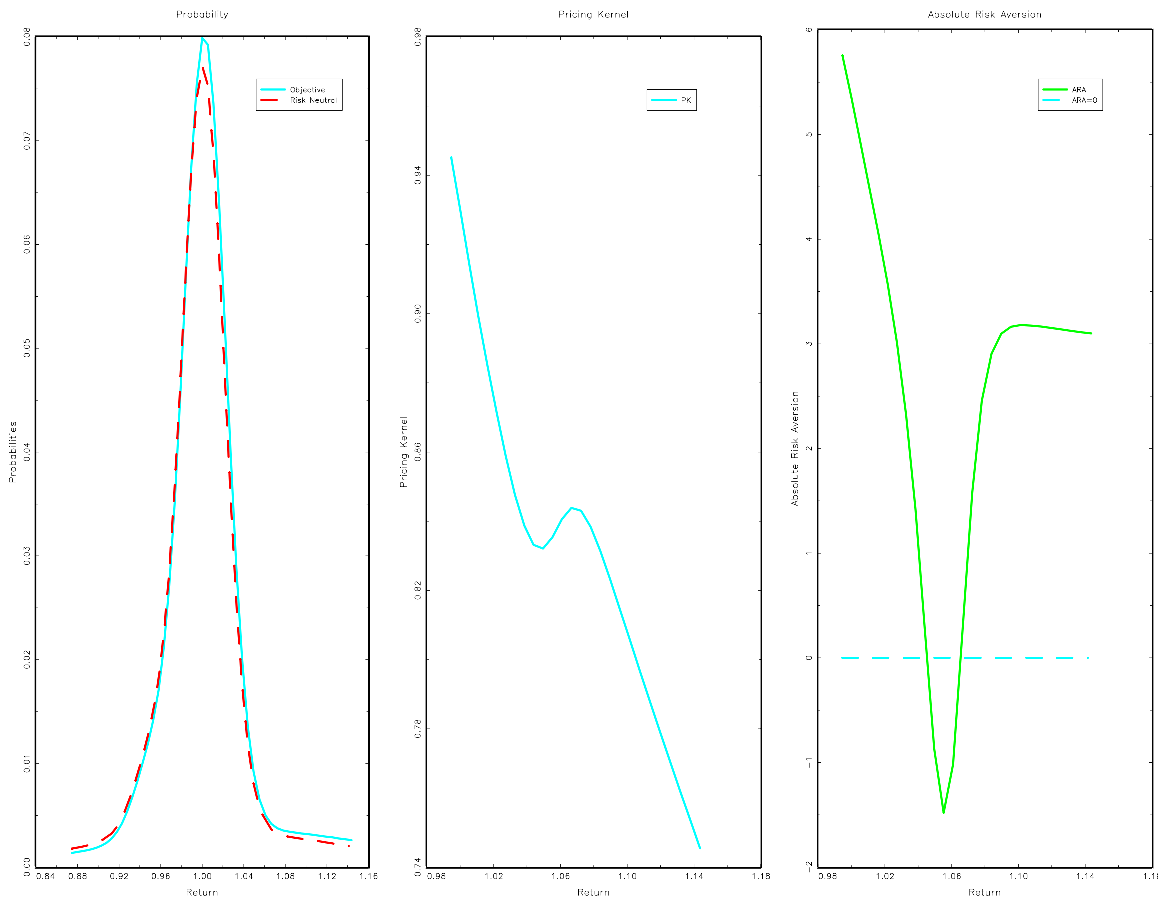

Figure 3: **Pricing Kernel (PK) and Absolute Risk Aversion (ARA) functions with state dependence in both preferences and fundamentals**. The preference parameters are $\beta = 0.95$, $\alpha = (-5, -3.5)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividend is 0.6. The number of wealth states is $n = 170$. The left-hand panel contains the marginal objective and risk-neutral distributions based on equations (11) and (15). The middle contains the corresponding pricing kernel (PK) function. The right-hand panel contains the absolute risk aversion (ARA) function across wealth states.

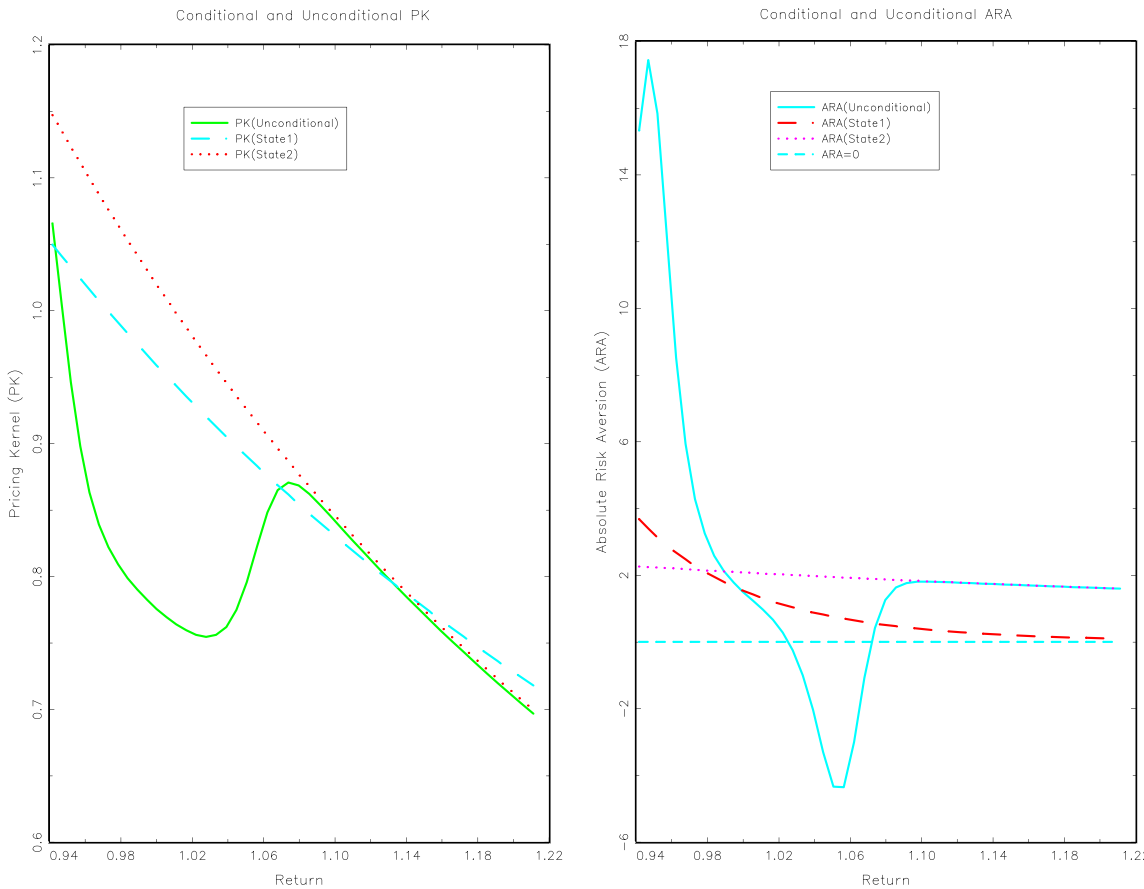

Figure 4: **Pricing Kernel (PK) and Absolute Risk Aversion (ARA) functions with state dependence in both preferences and fundamentals**. The preference parameters are $\beta = 0.95$, $\alpha = (-5, -3.5)$, $\rho = -10$. The regime probabilities are $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0009)$ and the standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0341)$. For the dividend rate, Y_{t+1}, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.02, 0.12)$. The correlation coefficient between consumption and dividend is 0.6. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional PK functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in equations (8) and (12). The unconditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in equations (9) and (13).

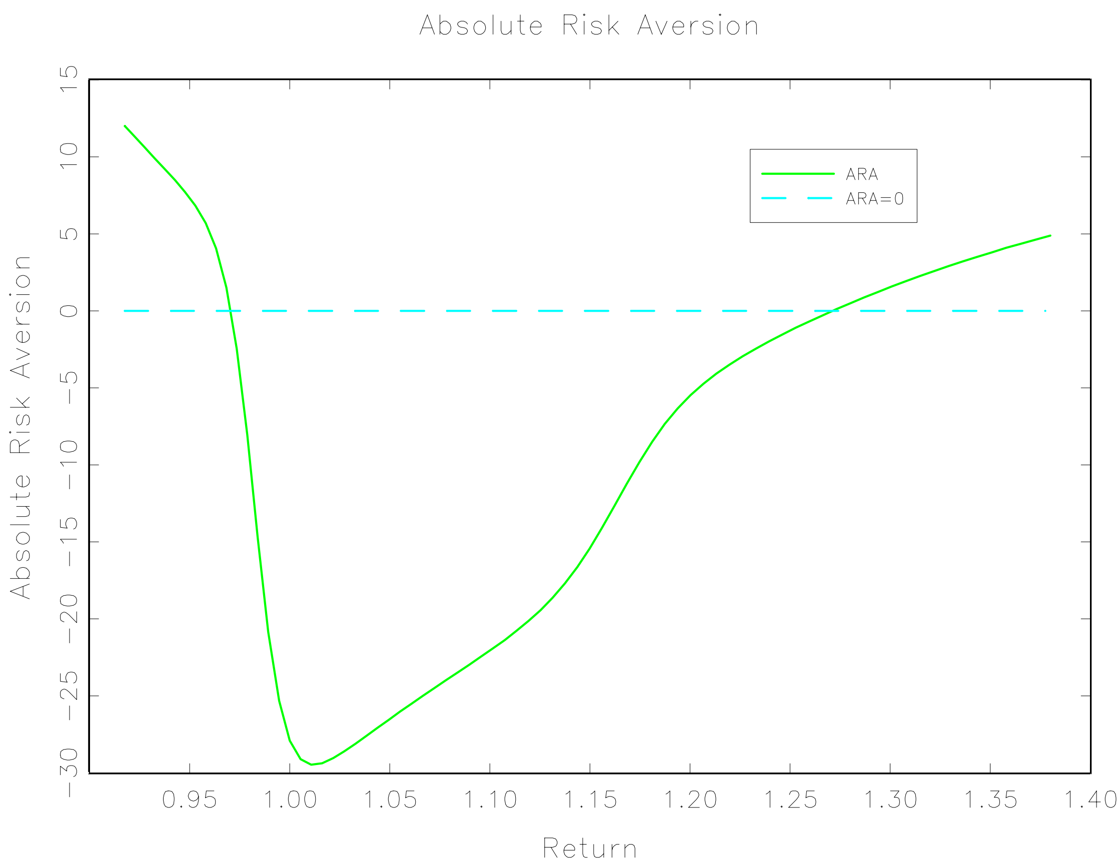

Figure 5: **Absolute Risk Aversion with dependence in fundamentals and beliefs**. The preference parameters are: $\phi = 0.03$, $\alpha = 10$, $\rho = (10, 0)$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0019)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0390)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.045, 0.20)$. The correlation coefficient between consumption and dividends is 0.45. The number of wealth states is $n = 170$.