Pricing and Hedging Options with Implied Asset Prices and Volatilities*

René Garcia

CIRANO and CIREQ, Université de Montréal

Richard Luger

Bank of Canada

Éric Renault

CIRANO and CIREQ, Université de Montréal

*Address for correspondence: René Garcia, Département de Sciences Économiques et CIREQ, Université de Montréal, C.P. 6128, Succ. Centre-Ville, Montréal, Québec, H3C 3J7, Canada.

The first two authors gratefully acknowledge financial support from the Fonds de la Formation de Chercheurs et l'Aide à la Recherche du Québec (FCAR), the Social Sciences and Humanities Research Council of Canada (SSHRC) and the MITACS Network of Centres of Excellence. The third author thanks CIRANO and CIREQ for financial support. We wish to thank Jean-Paul Minh Truong for research assistance. The views expressed in this paper are those of the authors. No responsibility for them should be attributed to the Bank of Canada.

Abstract

This paper proposes a class of empirical methodologies for pricing and hedging options based on implied asset prices and volatilities. For a given set of option prices written on the same asset, the methods simply invert the usual Black-Scholes formula to find implied asset price and volatility functions which satisfy theoretically-rationalized constraints on the shape of the option's payoff function and on the shape of the implied volatility smile. The out-of-sample performances of these new methods are compared with both simulated and actual option data. The methods proposed in this paper are found to have significantly smaller dollar pricing errors compared to the performance of the so-called practitioner's Black-Scholes model which uses only unconstrained implied volatilities.

1. Introduction

In applying option pricing models, practitioners encounter the difficulty that some of the parameters may be unobservable. In the benchmark Black and Scholes (1973) model (BS model hereafter), the volatility of the underlying stock price is not directly observed and therefore needs to be "estimated". While there are several statistical methods to estimate the underlying asset's price volatility, an alternative and common approach consists of extracting the volatility implied by an option price observed in the market. While the theoretical BS model implies that one option should be sufficient to recover the volatility parameter, it is common practice to extract in a given day as many volatilities as there are option prices available. For a given set of option prices π_{it} with strike prices K_{it} written on the same asset, BS implied volatilities σ_{it}^* are defined through:

$$\pi_{it} = BS(S_t, K_{it}, \sigma_{it}^*),$$

where S_t is the price of the underlying asset observed at time t. It is also typical to estimate a smooth relation by regressing the implied volatilities on a measure of "moneyness" and on time to maturity as in Dumas, Fleming and Whaley (1998). The fitted relation, often called the volatility smile, is then used for subsequent option pricing and hedging.

Another strand of the literature suggests to use the BS option pricing model to calculate jointly implied stock prices and implied volatilities. Manaster and Rendleman (1982) and Longstaff (1995) inverted the BS model to estimate both implied asset prices and volatilities and found that asset prices implicit in option premia contain information about the equilibrium asset price which is not fully reflected in the observed asset price. To explain these results, they argue that, in the presence of transaction costs or other market frictions, the asset price implicit in option premia may differ from the price observed in the market for the asset. While their methodology consists in extracting average implied stock price and volatility based on a set of options, one could exploit better the information contained in the set of implied stock prices and volatilities obtained from a given set of observed option prices

This paper provides a simple and parsimonious parametric methodology to exploit the information contained in implied stock prices and volatilities to construct better pricing and hedging strategies. Given a set of option prices written on the same asset, the method simply inverts the BS model to find implied asset prices and volatility functions which satisfy constraints on the shape of the option's payoff function and on the shape of the implied volatility smile. The approaches proposed by Manaster and Rendleman (1982) and Longstaff (1995) compute estimates of the implied parameters by minimizing the sum of squared deviations between predicted and actual option prices for a set of options. Such approaches, however, appear too global to capture not only the well-documented implied volatility smile but also the structure of option prices with respect to implied asset prices.

In a first section, we provide generalized BS models which can rationalize the fact that the implied stock price differs in equilibrium from the observed stock price. In these models, volatility will be stochastic. This randomness will also rationalize the observed volatility smile. A main ingredient of these models is the presence of an unobserved state variable which makes the option price an expectation with respect to this source of randomness. An important result is that a stochastic scaling factor of the stock price appears in the expectation. In Fouque, Papanicolaou and Sircar (2000), where the unobserved state variable is the volatility, the scaling factor is the result of a volatility risk premium and of a leverage effect coefficient ρ between the innovations in returns and volatility. In Garcia, Luger and Renault (2003), where the state variable affects the consumption and dividend processes in an intertemporal option pricing model, the stochastic scaling factor of the stock price includes the stochastic means, variances and covariances of the consumption and dividend processes. Our procedure allows to recover in a deterministic fashion these stochastic scaling factors which make the difference between the implied and the observed stock prices.

To assess the out-of-sample performance of these new methods and better understand the sources of the potential improvements over the traditional so-called practitioners' BS methodology, which will be one of our benchmarks, we simulate option prices in the jump-diffusion models estimated recently by Pan (2002). Thanks to several of these models we will illustrate how the presence of risk premia will be reflected in the performance differences between our methodology, which captures the distortions of the implied price and volatility, and simpler methodologies which do not. This simulated environment will help to point out the practical cases in which our methodology is most likely to lead to a performance improvement.

The second empirical exercise consists in applying our methodology to actual S&P 500 index option data. We separate our set of option prices into several categories of moneyness and time to maturity and conduct our experiment over nine years of daily data. We find significantly smaller dollar pricing errors compared to the performance of the practitioners' BS model which uses only implied volatilities.

The remainder of this paper is organized as follows. Section 2 reviews generalized BS models which serve as a theoretical background to the new empirical methodology we propose. Section 3 assesses the empirical performance of such an approach for simulated and observed option price data. Section 4 concludes.

2. Implied Asset Prices and Volatilities

In this section we will start by recalling theoretical models which generalize the basic BS model. The most popular extensions of the BS model is the stochastic volatility models of Hull and White (1987) and Heston(1993). We will see below that if the risk of volatility is priced and if innovations in returns are correlated with innovations in volatility, the stock price in the pricing formula will be adjusted by a scaling factor. We will also provide an example of an intertemporal equilibrium option pricing model proposed by Garcia, Luger and Renault (2003) where the stock price is scaled by a stochastic factor in a generalized BS pricing formula. These models provide a rationale for using empirical methodologies to extract from observed option prices not only implied volatilities, as it is currently the practice, but also implied stock prices. We develop therefore a new approach that will provide practitioners with a way to improve upon the current so-called practitioners' Black-Scholes methodology, which is a procedure based solely on implied volatilities.

2.1 Generalized Black-Scholes Models

The canonical stochastic volatility model in Heston (1993) is given by:

$$\begin{pmatrix} dY_t \\ dV_t \end{pmatrix} = \begin{pmatrix} \mu \\ \kappa(\theta - V_t) \end{pmatrix} dt + \sqrt{V_t} \begin{pmatrix} 1 & 0 \\ \rho\sigma_v & \sqrt{(1-\rho^2)}\sigma_v \end{pmatrix} dW_t \tag{1}$$

where Y_t is the logarithm of the stock price S_t, and $W_t = (W_{1t}, W_{2t})'$ is a vector of independent standard Brownian motions. Fouque, Papanicolaou and Sircar (2000) show that the price of a European call option can be written as:

$$\Pi_t = E_t \left[BS \left(S_t \xi_{t,T}, \bar{\sigma}_{t,T}^2 \right) \right] \tag{2}$$

where:

$$\bar{\sigma}_{t,T}^2 = \left(1 - \rho^2\right) \int_t^T V_\tau d\tau$$

and:

$$\xi_{t,T} = \exp \left[\int_t^T \left[a\left(V_\tau\right) + \rho\sqrt{V_\tau} \right] dW_{2\tau} \right] \exp \left[-\frac{1}{2} \int_t^T \left\{ a\left(V_\tau\right) + \rho\sqrt{V_\tau} \right\}^2 d\tau \right] \tag{3}$$

In other words, the price is given by the expectation of a BS-like option pricing formula with $S_t\xi_{t,T}$ instead of S_t. The scaling factor $\xi_{t,T}$ is stochastic since it is a function of V_t and contains both a volatility risk premium $a\left(V_\tau\right)$ and a leverage effect $\rho\sqrt{V_\tau}$. Moreover, one can see that it is equal to 1 in expectation.

Garcia, Luger and Renault (2003) develop a formula for the valuation of options written on stocks in an intertemporal equilibrium model. They assume that the logarithm of the consumption growth $X_t = Log\frac{C_t}{C_{t-1}}$ and of the dividend growth $Y_t = Log\frac{D_t}{D_{t-1}}$ follow stochastic processes where the mean, variance and covariance coefficients are function of latent state variables U. They assume that these state variables are exogenous and stationary and subsume all temporal links between the variables of interest (X_t, Y_t).With an additional normality assumption on the probability distribution of the fundamentals X and Y conditional on the state variables U :

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} |U_t^{t+1} \sim \aleph \left[\begin{pmatrix} m_{Xt+1} \\ m_{Yt+1} \end{pmatrix}, \begin{bmatrix} \sigma_{Xt+1}^2 & \sigma_{XYt+1} \\ \sigma_{XYt+1} & \sigma_{Yt+1}^2 \end{bmatrix} \right],$$

where the means and variance-covariance functions $m_{Xt+1}, m_{Yt+1}, \sigma^2_{Xt+1}, \sigma_{XYt+1}$ and σ^2_{Yt+1} are time-invariant and measurable functions with respect to U_t^{t+1}, which includes both U_t and U_{t+1}, .they obtain the following formula for the price of a European call option:

$$\pi_t = E_t \left\{ S_t Q_{XY}(t,T)\Phi(d_1) - K\widetilde{B}(t,T)\Phi(d_2) \right\}, \tag{4}$$

where:

$$d_1 = \frac{Log\left[\frac{S_t Q_{XY}(t,T)}{K\widetilde{B}(t,T)}\right]}{(\sum_{\tau=t+1}^{T} \sigma^2_{Y\tau})^{1/2}} + \frac{1}{2}(\sum_{\tau=t+1}^{T} \sigma^2_{Y\tau})^{1/2} \text{ and } d_2 = d_1 - (\sum_{\tau=t+1}^{T} \sigma^2_{Y\tau})^{1/2}.$$

The terms $Q_{XY}(t,T)$ and $\widetilde{B}(t,T)$ are stochastic factors which determine the price of the stock and the time t price of a bond which delivers one unit of the good at time T. They depend on the means and variance-covariance functions, themselves dependent on the latent variables, and on preference parameters.[1] It should be noticed that if $Q_{XY}(t,T) = 1$ and $\widetilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1)$, the option price (4) is nothing but the conditional expectation of the Black-Scholes price,[2] where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\overline{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\overline{\sigma}_{t,T} = \sqrt{\sum_{\tau=t+1}^{T} \sigma^2_{Y\tau}}$. This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and

[1] For their exact expressions and a proof of the formula, see Garcia, Luger and Renault (2003).

[2] We refer here to a BS option pricing formula where dividend flows arrive during the lifetime of the option and are accounted for in the definition of the risk neutral probability, while the option payoff does not include dividends. In other words, the BS option price is given by:

$$\begin{aligned} \pi_t^{BS} &= e^{-r(T-t)} E_t[Max(0, S_T - K)] \quad (5) \\ &= e^{-\delta(T-t)} S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (6) \end{aligned}$$

since in the risk neutral world:

$$Log \frac{S_T}{S_t} \rightsquigarrow \mathcal{N}((r-\delta)(T-t), \sigma^2(T-t)), \tag{7}$$

where δ is the intensity of the dividend flow.

White (1987) stochastic volatility extension, since $\overline{\sigma}_{t,T}^2 = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the cumulated volatility $\int_t^T \sigma_u^2 du$ in the Hull-White continuous-time setting. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). As in the case above, we have the property that $E_t\left[Q_{XY}(t,T)\right] = 1$ by the stock pricing formula.

In the next section we will develop an empirical methodology which assume away the stochastic nature of the scaling factors or the volatilities as the practitioners do when they extract implied volatilities based on the BS formula.

2.2 A Practitioner's Approach

The generalized BS models just seen appear to be naturally amenable to an empirical methodology where implied asset prices and implied volatilities are simultaneously extracted from option prices. Indeed, for a given maturity, we can simultaneously define an implied price and volatility as:

$$\pi_{it} = BS(S_{it}^*, K_{it}, \sigma_{it}^*),$$

for any pair $i = 1, 2$ of strike prices K_{it} observed at time t. The variables S_{it}^* correspond to the scaled stock prices illustrated in the previous section. Such an empirical approach is justified by the following result.

Proposition. *If $K_1 \neq K_2$ then the mapping:*

$$\begin{pmatrix} S \\ \sigma \end{pmatrix} \to \begin{pmatrix} BS_{K_1}(s,\sigma) \\ BS_{K_2}(s,\sigma) \end{pmatrix}$$

is locally invertible in a neighborhood of (S, σ) for any positive S and σ.

A proof is provided in the Appendix. Thus, when one considers at the same time two option contracts written on the same asset with two different strike prices, the two option prices, if conformable to the Black-Scholes model, allow one to extract simultaneously an implied volatility parameter and an implied asset price as well.

We can therefore build a class of empirical methodologies as in Manaster and Rendleman (1982) and Longstaff (1995), who invert the Black and Scholes model to estimate both the implied asset prices and the implied volatilities by minimizing the sum of squared deviations between predicted and actual option prices. Those approaches, however, appear too global to capture not only the well-documented volatility smile but also the structure of option prices with respect to implied asset prices.

The class of empirical methodologies that we propose consists of inverting the usual Black-Scholes formula to find implied asset price and volatility *functions*. Specifically, the methodologies are defined by

$$\min_{\theta,\gamma} \sum_i \left(\pi_{it} - BS\left(f(x_{it}, \tau_i; \theta)S_t,\ K_{it},\ g(x_{it}, \tau_i; \gamma)\right)\right)^2, \tag{8}$$

where the summation is over a given set of call options π_{it} with strike prices K_{it} written at time t on S_t, for a given maturity T_i, with $\tau_i = T_i - t$.The moneyness of the option is measured by $x_{it} = \log \frac{S_t}{K_{it}B(t,T_i)}$. Therefore, the call option is in the money if $x_{it} > 0$, out of the money if $x_{it} < 0$ and at the money if $x_{it} = 0$. Starting from the generalized BS models summarized in the previous section, we specify a parametric form for the functions $f(\cdot)$ and $g(\cdot)$ in (8) in order to capture the main features of these pricing models in a parsimonious and user-friendly way. To introduce the characteristics imposed on these functions, we start with a simpler case where the function $f(.)$ is constrained to be identically equal to one. This case is suggested by a Hull and White (1987) type of option pricing model where the option price is simply viewed as:

$$\pi_t = \widetilde{E}_t[BS(S_t, K, \widetilde{\sigma}). \tag{9}$$

The expectation operator $\widetilde{E}$ is applied over some heterogeneity distribution of the volatility parameter $\widetilde{\sigma}$.[3] If option prices obey (9), then Renault and Touzi (1996) and Renault (1997) have shown that computed volatility smiles are U-shaped, minimum at-the-money (whit the measure of moneyness just defined) and symmetric, i.e. $\sigma^*(x_{it}) = \sigma^*(-x_{it})$. Therefore, the same level of implied volatility will be obtained from two call options for which the strike prices have a geometric mean equal to the forward stock price. With this theoretical justification in mind

[3] As discussed by Renault (1997), this heterogeneity distribution can be produced by several types of jumps or stochastic volatility processes, insofar as leverage or volatility feedback effects are excluded.

we will refer to this case as the HW case. In this particular case, we will ignore any implied underlying asset price and just compute an implied volatility function:

$$\sigma^*(x) = g(x, \tau; \gamma) \tag{10}$$

To impose the U-shape just referred to, we choose a parametric specification of the function $g(x, \tau; \gamma)$ which obeys the following conditions:

$$\begin{aligned} g(x, \tau; \gamma) &= g(-x, \tau; \gamma) \\ x\frac{\partial g}{\partial x}(x, \tau; \gamma) &> 0 \text{ for } x \neq 0 \\ \frac{\partial g}{\partial x}(x, \tau; \gamma) &\mid_{x=0} = 0 \end{aligned} \tag{11}$$

Of course, it is by now well-known that observed volatility smiles are mostly asymmetric. We argue that this asymmetry will come about by using a nonconstant function f to introduce an implied stock price instead of the observed stock price when computing BS implied volatilities. Indeed, Renault (1997) has shown by simulation that a little discrepancy between these two prices will produce substantial asymmetries. Moreover, the generalized BS models of the previous section suggest that the discrepancy is produced by a Jensen effect, since the underlying stock price is simply multiplied by a factor Q of expectation unity. Given these considerations, the function $f(.)$ will be all the more different from one that the BS function is convex with respect to the underlying stock price. It is known (see Hull, 2001) that the measure of this convexity, $\Gamma = \frac{\partial^2 BS}{\partial S^2}$, is maximal at the money and decreases towards zero further away from the money. Therefore, we will choose a parametric specification of the function f which obeys the following conditions:

$$\begin{aligned} f(x, \tau; \gamma) &= f(-x, \tau; \gamma) \\ x\frac{\partial f}{\partial x}(x, \tau; \gamma) &> 0 \text{ for } x \neq 0 \\ \frac{\partial f}{\partial x}(x, \tau; \gamma) &\mid_{x=0} = 0 \\ \lim_{|x|=\infty} f(x, \tau; \gamma) &= 1 \end{aligned} \tag{12}$$

We will therefore base our empirical procedure on the minimization program (8) where the functions f and g will be given by:

$$f(x_{it}, \tau_i; \theta_1, \theta_2) = \frac{1}{1 - \theta_1 \exp(-\theta_2 |x_{it}|)}, \tag{13}$$

where $\theta_1 \in [0, 1[$ and $\theta_2 \geq 0$, and

$$g(x_{it}, \tau_i; \gamma_1, \gamma_2) = \gamma_1 \exp(\gamma_2 |x_{it}|), \tag{14}$$

where $\gamma_1, \gamma_2 \geq 0$. Note that these parsimonious functions satisfy the sets of conditions we just established.We do not account directly for a maturity effect. In fact, the theoretical generalized BS models in the previous section suggest that the importance of the two aforementioned effects (the presence of a volatility smile and its asymmetry) decreases with time to maturity.[4] We will investigate in the empirical section if time to maturity should be introduced in the functions f and g to capture these effects.

In our empirical procedure, which we will denote GBS in reference to its theoretical underpinnings described above, we will also consider the HW case aforementioned where $f(\cdot)$ is restricted to be identically one to assess the part of the performance which is due to the asymmetric correction through an implied price. In both cases, the procedure will consist in estimating the set of parameters (θ_1, θ_2, γ_1, γ_2) with all the options traded in one day and used these estimated functions in the BS model to get an empirical pricing function for the next day. We consider that the daily interest rate is observed and use $\exp(-r_t \tau_i)$ for discounting.

3. Performance Comparisons

The practitioners' Black-Scholes pricing procedure serves as the benchmark for comparisons since it is mostly used in practice. This procedure is implemented by first finding the volatilities implied by a given set of call options. Next, the implied volatilities are regressed on a constant, moneyness, and moneyness squared.

[4]In the limit, as the time to maturity tends to infinity, the two effects will vanish. This implied that the functions f and g should be such that $f(x, \tau; \gamma)$ is a decreasing function of τ with $\lim_{\tau=\infty} f(x, \tau; \gamma) = 1$ and $|\frac{\partial g}{\partial x}(x, \tau; \gamma)|$ is a decreasing function of τ with $\lim_{\tau=\infty} |\frac{\partial g}{\partial x}(x, \tau; \gamma)| = 0$. This can be achieved by dividing θ_2 and γ_2 by the time to maturity of the option.

Finally, the fitted values are used in the Black-Scholes model to get an empirical pricing function.

To assess the performance of the empirical procedure described in the previous section with respect to the benchmark we proceed first by simulation and then we use actual data. The simulation exercise will be based on a jump-diffusion model proposed by Bates (2000) and estimated by Pan (2002). This model will be useful to understand the role played by the various risk premia in the price of the option and the success achieved by our procedure in capturing the corresponding asymmetries Pan (2002) argues that jump risk premia are essential to explain the asymmetric nature of the volatility smile.

The data we will employ are daily S&P 500 index European style call options from the Chicago Board Options Exchange covering the period January 1988 to December 1996. The S&P 500 index option market is extremely liquid and it is one of the most active option markets in the United States. In many respects, this market is the closest to the theoretical setting of the Black-Scholes model.

For both our simulation and actual results, we will report the average absolute pricing errors for three maturity categories (less than 60 days, between 60 and 180 days, and more than 180 days) and six moneyness categories defined by: [1] : $x \leq -0.06$, [2] : $-0.06 < x \leq -0.03$, [3] : $-0.03 < x \leq 0$, [4] : $0 < x \leq 0.03$, [5] : $0.03 < x \leq 0.06$, and [6] : $0.06 < x$. The moneyness categories are defined from most "out-of-the-money" ($x < 0$) to most "in-the-money" ($x > 0$).

3.1 Pricing and hedging simulated options

Before comparing pricing and hedging performances with the actual data, we compare the empirical methodologies on artificially created data sets. Following Pan (2002), we adopt the Bates (2000) model to characterize stock return dynamics. That model introduces three sources of uncertainty to the underlying price dynamics: (i) diffuse price shocks, (ii) diffuse volatility shocks, and (iii) price jumps. Assuming a non-dividend-paying stock S, its return dynamics are described by

$$\mathrm{d}S_t = \left[r_t - \eta^S V_t + \lambda V_t(\mu - \mu^*)\right] S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^{(1)} + \mathrm{d}Z_t - \mu S_t \lambda V_t \mathrm{d}t, \quad (15)$$

$$\mathrm{d}V_t = \kappa_v(\bar{v} - V_t)\mathrm{d}t + \sigma_v\sqrt{V_t}\left(\rho\mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\mathrm{d}W_t^{(2)}\right), \tag{16}$$

where:

- r_t is the instantaneous spot interest rate;
- V_t is the diffusion component of return variance whose dynamics are described in (16) by a one-factor square-root process with constant long-run mean $\bar{v}$, mean-reversion rate κ_v, and variance coefficient σ_v;
- $W_t^{(1)}$ and $W_t^{(2)}$ are each standard Brownian motion with $\mathrm{Cov}_t[\mathrm{d}W_t^{(1)}, \mathrm{d}W_t^{(2)}] = \rho\mathrm{d}t$
- Z_t is a Poisson counter with instantaneous state-dependent stochastic intensity λV_t such that $\Pr(\mathrm{d}Z_t = 1) = \lambda V_t\mathrm{d}t$ for some non-negative constant λ. Jump sizes are lognormally, identically and independently distributed over time, with mean relative jump size $\mu = \exp(\mu_J + \sigma_J^2/2) - 1$.
- The stock return drift component depends on two risk-premium components: $\eta^S V_t$ and $\lambda V_t(\mu - \mu^*)$, which are associated with the premia for "Brownian" return risks and jump risks, respectively (see Pan 2002, Section 2.2).

The autonomous stochastic spot interest-rate process is a single-factor square-root model of the Cox, Ingersoll, and Ross (1985) type defined by

$$\mathrm{d}r_t = \kappa_r(\bar{r} - r_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}W_t^{(r)}, \tag{17}$$

where κ_r, $\bar{r}$, and σ_r are respectively the mean-reversion rate, long-run mean, and volatility coefficient.

With this setting, in which markets are incomplete, Pan (2002) derives a risk-adjusted option pricing equation under a risk-neutral measure Q of the form:

$$C_t = E_t^Q\left[\exp\left(\int_t^T r_u\mathrm{d}u\right)\max(0, S_T - K)\right], \tag{18}$$

and she provides a numerical integration method to compute the option price under the risk-neutral dynamics of S and V.

Pan (2002) argues that the risk-premium component contained in option prices may have a significant role in reconciling the dynamics of the underlying asset price implicit in the options market with those observed on the actual market of the underlying asset. In particular, she finds that jump-risk premia are important factors explaining volatility "smirks". In order to examine the role of risk premia on our empirical methodology, we consider four different versions of Pan's model, where different risk-premium structures: jump-risk premia (SVJ0), volatility-risk premia (SV), no risk premia (SV0), and both jump and volatility risk premia (SVJ), play a role in option pricing.

Using the estimated parameter values for each of these models reported by Pan (2002, Tables 1 and 3), we simulated 1000 trading days as follows. First, a moneyness and a maturity category were randomly selected according to the empirical marginal probabilities of occurrence observed in the actual data. In the actual daily data, we find that over a nine-year period on average each year nearly 55 per cent of call options have maturities less than 60 days, 35 per cent between 60 and 180 days, and 10 per cent more than 180 days. Over that period, the average occurrence of moneyness each year is nearly symmetric around zero with 50 per cent falling in moneyness category [3] and [5], 30 per cent in [2] and [6], and 20 percent in [1] and [7]. The actual moneyness x_i and maturities τ_j where then drawn from uniform distributions over the selected moneyness and maturity categories. Five moneyness and five maturity values were selected this way. Given a stock price and a value of the spot interest rate, we then obtain 25 strike prices from $K_{i,j} = S_t \exp(r_t \tau_j - x_i))$, $i, j = 1, ..., 5$. The initial stock price was set at 100\$ and the initial per annum interest rate at 5%. For a given set of parameter values for the joint process (S, V, r) specified in (15)-(16) and (17), each of the resulting 25 options were priced à la Pan each simulated day. Whenever an option arrived within a week of expiration, it was replaced by a newly created one following the same procedure for moneyness and maturity selection.

Given these panels of simulated data, we rely on the previous day's option prices to back out the required parameter values and then use them as inputs to compute the current day's option prices based on the practitioners' models. Next, we subtract the model-determined price from its observed (simulated) counterpart to compute the absolute pricing error. This procedure is repeated for the 25 call options traded over the thousand days in the simulation experiment. These steps are separately followed for Pan's SVJ0, SV, SV0, and SVJ model as data generating processes (DGP).

Table 1 reports the average absolute pricing errors for the various maturity and moneyness categories [1]-[6]. First note that GBS always performs better than BS for medium and long term options. As the maturity horizon gets longer it appears that the performance of BS deteriorates dramatically relative to GBS, especially in the presence of price jump risks. For short-term options, BS performs uniformly better than GBS and HW when the only risk premium is that for volatility shocks (model SV). Intuitively, the equilibrium and implied asset prices coincide in this case such that any attempt to find a "corrected" price results in overfitting. Interestingly though, when there are no risk premia at all (model SV0) the GBS procedure yields smaller pricing errors relative to the BS procedure for in-the-money options. It appears from Table 1 that GBS can offer important pricing gains, especially when option prices contain a jump-risk premium.

We next consider hedging performance. We imagine a financial institution that wishes to hedge a short position in a call option with $T-t$ days to expiration and strike price K. Let $\pi_{t,T-t}$ denote the time-t price of that option. At time t, the financial institution makes up a portfolio by selling the call option and simultaneously purchasing Δ_t shares of the stock. The time-t residual cash for the hedge $X_t = \pi_{t,T-t} - \Delta_t S_t$ is invested in an instantaneously maturing risk-free bond. Suppose the portfolio is rebalanced every day. At time $t+1$, the hedging error is

$$\varepsilon_{t+1} = \Delta_t S_{t+1} + X_t \exp(r_t) - \pi_{t+1,T-t-1}.$$

Given the panels of simulated data, the hedging errors were obtained as follows. First, the set of parameter values implied by all 25 call options on day $t-1$ were obtained. Then, on day t, the hedge described above is constructed using those implied parameter values. For the ad hoc Black-Scholes procedure, the delta (the derivative of the pricing function with respect to the stock price) is computed as

$$\Delta(S_t, \sigma) = \Phi\left(\frac{x_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right), \tag{19}$$

where time $t-1$ implied volatilities are used for σ. Hedge deltas based on the generalized BS are computed as

$$\Delta\left(\tilde{S}_t, g(x_t); \hat{\gamma}_1, \hat{\gamma}_2)\right) f(x_t; \hat{\theta}_1, \hat{\theta}_2),$$

where $\tilde{S}_t = f(x_t; \hat{\theta}_1, \hat{\theta}_2) S_t$ and $\hat{\theta}$ and $\hat{\gamma}$ are estimated at time $t-1$. For the particular case à la "Hull-White", hedge deltas are computed as in (19) with the original stock price S_t and corrected volatilities $g(x_t); \hat{\gamma}_1, \hat{\gamma}_2)$ in lieu of σ.

Finally, assuming that the hedge is liquidated the following day, we compute the date $t+1$ hedging error. These steps are repeated for each option and every trading day in the sample thereby obtaining the absolute hedging errors reported in Table 2.

Besides for short-term in-the-money options, where there are some small differences between hedging based on GBS relative to BS, it appears from Table 2 that each model has roughly the same hedging performance. A plausible explanation is that the magnitude of the error stays the same between three consecutive days for all models and therefore eliminate itself.

3.2 Pricing and hedging actual options

Tables 4-6 report the absolute one-day-ahead pricing errors for the three empirical pricing procedures GBS, HW, and BS, respectively, while the number of actual S&P 500 index options priced is found in Table 3. These results are based on the same procedure described above where only the previous day's cross-section of options is used to get parameter estimates.

As with the simulated data, we see that GBS offers important pricing gains relative to BS. This is especially true as the maturity becomes longer. For many of the longer term options, the BS mean absolute pricing errors are more than double those of GBS. The bad performance of the practitioners' BS procedure is that the coefficients of the linear regression are good to fit the smile at a short horizon, where most of the data are, but at a long horizon, where data are scarcer. The properties imposed on our functions add constraints for the estimation and help when the number of data points is small.

Even for short-term options, GBS outperforms BS albeit by a narrower margin.

The more into the money the option is (categories [4] to [6]), the greater are the pricing gains from GBS relative to BS.

Except for some small differences for short-term near-the-money options, GBS outperforms the HW specification. Comparing the overall pricing performance of GBS with that of HW it becomes clear that using a "corrected" price offers gains beyond what a "corrected" volatility smile can offer.

We have also investigated the performance of the three procedures when time to maturity is specifically taken into account. We add time to maturity to the regression for the practitioners' BS model and we divide the parameters θ_2 and γ_2 by the time to maturity of the option. The results we obtain are very similar to the previous results.

Comparing the results in Tables 7-9, we see that the relative hedging performances of the three empirical pricing procedures does not reveal as strong differences as in the pricing performances. There appears to be some small gains to be made from using GBS or HW over BS in terms of hedging in some cases. However, as with the simulations, it is obvious that the most important gains to be made from using the empirical methodologies proposed in this paper are not so much in terms of hedging but rather in terms of pricing.

4. Conclusion

In this paper, we proposed a class of empirical methodologies that extend the usual practice of using the Black-Scholes model with implied volatilities. By estimating implied asset and volatility functions which satisfy some theoretically-rationalized constraints, we found that significant dollar gains can be made compared to the practitioners' Black-Scholes model which uses only unconstrained implied volatilities. While the new methods do not necessarily provide improvements in terms of delta-hedging, the pricing gains from the new methods can be significant, especially when option prices contain a jump-risk premium component.

Finally, it should be emphasized that from a numerical point of view, the methods proposed in this paper are as easy and quick as solving a non-linear least squares problem. Any computer software which can minimize a quadratic form can readily handle the methods proposed here.

Appendix

Proof of Proposition

We have to show that the Jacobian matrix of the mapping of interest is non-singular. But the gradient of the mapping: $(S, \sigma) \rightarrow BS_K(S, \sigma)$ is the vector of the coefficients delta and vega of the call, that is (see e.g. Hull (1993), pages 314 and 329):

$$[\phi(d_K), S_t\sqrt{T-t}\varphi(d_K)]$$

where $\varphi(d) = \phi'(d)$ is the density function of the standard normal and:
$d_K = \log \frac{S_t}{KB(t,T)} + \sigma\sqrt{T-t}$.

In other words, we have to show that for $K_1 \neq K_2$, the two vectors:

$$[\phi(d_{K1}), S_t\sqrt{T-t}\varphi(d_{K1})]$$
$$[\phi(d_{K2}), S_t\sqrt{T-t}\varphi(d_{K2})]$$

are linearly independent, that is:

$$\frac{\varphi(d_{K1})}{\phi(d_{K1})} \neq \frac{\varphi(d_{K2})}{\phi(d_{K2})}.$$

This is a direct consequence of the strict monotonicity of the Mills ratio:

$$x \rightarrow f(x) = \frac{\varphi(x)}{\phi(x)}.$$

Actually:

$$f'(x) = -\frac{\varphi(x)}{\phi^2(x)}[x\phi(x) + \varphi(x)] < 0$$

since $g(x) = x\phi(x) + \varphi(x) > 0$, given that: $g'(x) = \phi(x) > 0$ and $\lim_{x\rightarrow-\infty} g(x) = 0$. QED.

References

[1] Amin, K.I. and R. Jarrow (1992), "Pricing Options in a Stochastic Interest Rate Economy," Mathematical Finance, 3(3), 1-21.

[2] Bates, D. (2000), "Post-'87 Crash Fears in S&P 500 Futures Options," Journal of Econometrics 94, 181-238.

[3] Black, F. 1975. "Fact and Fantasy in the Use of Options," Financial Analysts Journal, 36-72.

[4] Black, F. and M. Scholes (1973), "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, 81, 637-659.

[5] Broadie, M., J. Detemple, E. Ghysels and O. Torres (2000), "American Options with Stochastic Dividends and Volatility: A Nonparametric Investigation," Journal of Econometrics, 94, 53-92.

[6] Cox, J.C.. Ingersoll, J.E., and S.A. Ross (1985), "A Theory of the Term Structure of Interest Rates," Econometrica 53, 385-407.

[7] Fouque J-P., G. Papanicolaou and K. R. Sircar (2000), Derivatives in Financial Markets With Stochastic Volatility, Princeton University Press.

[8] Garcia, R., R. Luger and E. Renault (2003), "Empirical Assessment of an Intertemporal Option Pricing Model with Latent Variables," forthcoming, Journal of Econometrics.

[9] Heston, S. (1993), "A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," The Review of Financial Studies 6, 327-343.

[10] Hull, J. and A. White (1987), The Pricing of Options on Assets with Stochastic Volatilities, Journal of Finance, 42, 281-300.

[11] Hull, J. (2001), Options, Futures, and Other Derivative Securities, Prentice Hall, 4th edition.

[12] Longstaff, F. A. (1995), "Option Pricing and the Martingale Restriction," Review of Financial Studies, 8, 1091-1124.

[13] Manaster, S. and R.J. Rendleman (1982), "Option Prices as Predictors of Equilibrium Stock Prices," Journal of Finance 37, 1043-1057.

[14] Merton, R. C. (1973), "Rational Theory of Option Pricing," Bell Journal of Economics and Management Science 4, 141-183.

[15] Pan, J. (2002), "The Jump-risk Premia Implicit in Options: Evidence from an Integrated Time-series Study," Journal of Financial Economics 63, 3-50.

[16] Renault, E. (1997), "Econometric Models of Option Pricing Errors," Advances in Economics and Econometrics, Kreps and Wallis Ed., Cambridge University Press.

[17] Renault, E. and N. Touzi (1996), "Option Hedging and Implied Volatilities in a Stochastic Volatility Model," Mathematical Finance, 6, 279-302.

[18] Turnbull, S., and F. Milne (1991), "A Simple Approach to Interest-Rate Option Pricing," Review of Financial Studies, 4, 87-121.

Table 1

Absolute Pricing Errors (in dollars): Simulated Data

DGP	Model	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
		[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
SVJ0	GBS	0.61	0.77	0.56	0.51	0.52	0.44	0.53	0.57	0.46	0.41	0.41	0.46	1.36	1.09	0.90	1.31	1.29	1.32
	HW	0.66	1.14	1.42	1.64	1.61	1.26	0.76	0.77	0.78	0.80	0.79	0.78	3.11	2.70	2.22	2.30	2.39	2.32
	BS	0.45	0.60	0.73	0.80	0.80	0.61	0.99	1.41	1.42	1.47	1.43	1.52	3.22	3.84	4.07	4.62	4.47	4.46
SV	GBS	0.32	0.50	1.26	0.84	0.35	0.14	0.32	0.30	0.35	0.32	0.33	0.18	0.49	0.94	1.21	1.54	1.19	0.66
	HW	0.35	0.52	0.50	0.42	0.45	0.22	0.44	0.33	0.34	0.36	0.40	0.62	0.61	1.07	1.22	1.41	1.20	1.57
	BS	0.12	0.24	0.25	0.22	0.21	0.09	0.33	0.41	0.42	0.45	0.44	0.39	1.06	1.59	1.83	2.01	1.75	1.03
SV0	GBS	0.51	0.64	0.39	0.30	0.34	0.27	0.43	0.42	0.33	0.26	0.30	0.29	0.93	0.91	0.94	0.86	0.87	1.04
	HW	0.50	0.73	0.92	0.96	0.86	0.56	0.51	0.49	0.50	0.48	0.51	0.43	1.77	1.73	1.68	1.44	1.46	1.61
	BS	0.49	0.49	0.52	0.48	0.54	0.46	0.99	0.91	0.89	0.93	0.94	0.92	1.68	1.69	2.86	2.61	1.90	1.12
SVJ	GBS	0.15	0.30	0.25	0.26	0.28	0.22	0.20	0.32	0.27	0.24	0.29	0.24	0.80	0.81	0.64	0.75	0.84	0.63
	HW	0.16	0.42	0.57	0.67	0.66	0.57	0.32	0.41	0.42	0.43	0.45	0.39	1.54	1.63	1.27	1.25	1.48	1.10
	BS	0.11	0.24	0.27	0.30	0.32	0.26	0.33	0.55	0.66	0.75	0.76	0.73	1.63	2.09	2.12	2.33	2.59	2.64

Table 2

Absolute Hedging Errors (in dollars): Simulated Data

DGP	Model	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
		[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
SVJ0	GBS	0.08	0.10	0.14	0.16	0.14	0.11	0.02	0.03	0.05	0.06	0.06	0.05	0.03	0.02	0.04	0.05	0.03	0.02
	HW	0.08	0.11	0.12	0.10	0.09	0.07	0.03	0.03	0.04	0.03	0.03	0.03	0.01	0.01	0.02	0.02	0.03	0.04
	BS	0.08	0.10	0.11	0.10	0.09	0.07	0.02	0.03	0.03	0.03	0.03	0.04	0.02	0.01	0.01	0.02	0.02	0.04
SV	GBS	0.08	0.14	0.17	0.18	0.15	0.06	0.07	0.09	0.09	0.10	0.10	0.08	0.09	0.11	0.12	0.13	0.10	0.06
	HW	0.09	0.15	0.16	0.17	0.16	0.08	0.09	0.09	0.10	0.11	0.10	0.11	0.10	0.11	0.12	0.13	0.10	0.11
	BS	0.08	0.15	0.15	0.16	0.15	0.06	0.08	0.09	0.10	0.10	0.10	0.07	0.10	0.11	0.12	0.12	0.10	0.07
SV0	GBS	0.05	0.07	0.12	0.14	0.13	0.08	0.02	0.03	0.04	0.05	0.04	0.03	0.02	0.03	0.04	0.04	0.02	0.01
	HW	0.06	0.08	0.08	0.07	0.08	0.05	0.02	0.02	0.03	0.03	0.03	0.02	0.01	0.01	0.02	0.03	0.03	0.02
	BS	0.06	0.08	0.08	0.07	0.07	0.05	0.03	0.02	0.02	0.02	0.03	0.02	0.03	0.02	0.01	0.02	0.03	0.02
SVJ	GBS	0.04	0.07	0.09	0.10	0.09	0.07	0.02	0.02	0.03	0.03	0.04	0.03	0.02	0.01	0.02	0.02	0.02	0.02
	HW	0.05	0.08	0.09	0.09	0.08	0.06	0.03	0.04	0.04	0.03	0.03	0.03	0.01	0.02	0.02	0.03	0.03	0.06
	BS	0.05	0.08	0.09	0.09	0.08	0.06	0.02	0.03	0.03	0.03	0.04	0.04	0.01	0.01	0.02	0.03	0.03	0.06

Table 3

Number of Options Priced

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
1988	293	401	661	618	366	232	442	231	269	291	197	175	95	37	43	54	43	64
1989	73	303	757	773	366	211	120	253	289	340	242	115	26	35	40	41	37	34
1990	294	484	828	802	427	197	425	366	423	423	285	264	94	57	66	62	60	66
1991	123	440	926	884	456	309	260	338	491	525	304	270	106	116	130	119	109	116
1992	70	383	953	947	430	140	169	464	652	661	295	135	73	96	111	94	82	81
1993	62	300	1001	1088	341	129	70	433	634	712	247	61	89	107	75	110	66	48
1994	26	488	1194	1150	414	95	240	703	836	793	234	95	185	177	186	170	100	91
1995	15	376	1330	1374	561	367	142	460	827	947	394	221	189	187	178	201	175	124
1996	310	1143	1746	1533	709	454	435	651	844	851	400	366	167	162	145	189	177	153

Table 4

Absolute Pricing Errors (in dollars): Generalized Black-Scholes Model

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.49	0.53	0.54	0.71	0.89	0.98	0.50	0.76	0.87	0.95	0.89	1.25	0.69	0.81	0.95	1.07	0.96	1.21
1989	0.53	0.55	0.58	0.74	0.95	0.85	0.55	0.59	0.75	0.85	0.80	0.97	0.57	0.80	0.80	1.02	1.23	1.90
1990	0.47	0.59	0.70	0.92	1.13	1.07	0.76	0.82	0.93	1.08	1.32	1.29	1.28	1.10	1.12	1.53	1.27	1.61
1991	0.49	0.54	0.63	0.77	0.98	1.15	0.69	0.72	0.86	0.99	1.23	1.12	0.90	0.93	1.05	1.62	1.61	1.48
1992	0.47	0.42	0.52	0.69	0.87	1.01	0.60	0.70	0.73	0.95	1.00	1.00	0.85	0.73	1.21	1.34	1.39	1.03
1993	0.47	0.42	0.52	0.60	0.72	0.97	0.51	0.56	0.67	0.83	0.95	0.68	0.77	0.81	1.18	1.30	1.21	1.44
1994	0.19	0.44	0.55	0.64	0.91	1.38	0.50	0.70	0.77	0.88	1.13	0.94	0.64	0.88	1.28	1.74	1.52	1.30
1995	0.24	0.31	0.44	0.68	0.97	1.15	0.36	0.49	0.68	0.87	1.11	1.09	0.79	0.79	1.07	1.21	1.28	1.21
1996	0.48	0.54	0.80	1.17	1.68	1.94	0.85	1.03	1.34	1.51	2.05	1.99	1.57	1.76	1.73	1.82	2.12	2.01

Table 5

Absolute Pricing Errors (in dollars): Hull-White Model

Year	Days to Expiration $\leq$ 60						60 $<$ Days to Expiration $\leq$ 180						180 $<$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.54	0.55	0.50	0.67	0.78	0.98	0.67	0.78	0.86	0.98	0.90	1.12	0.82	0.89	1.07	1.20	1.04	1.27
1989	0.66	0.60	0.56	0.68	0.90	0.84	0.67	0.78	0.78	0.84	0.80	0.92	0.54	0.93	0.88	1.10	1.30	1.77
1990	0.48	0.60	0.64	0.89	1.10	1.10	1.10	1.00	0.97	1.14	1.38	1.32	2.06	1.50	1.18	1.54	1.35	1.75
1991	0.51	0.56	0.62	0.73	0.91	1.18	0.92	0.84	0.90	1.03	1.22	1.08	1.30	1.27	1.21	1.67	1.65	1.49
1992	0.46	0.44	0.51	0.61	0.75	1.01	0.71	0.77	0.74	0.95	0.98	0.88	1.02	0.87	1.20	1.36	1.44	1.06
1993	0.46	0.44	0.50	0.56	0.73	1.08	0.61	0.66	0.69	0.82	0.91	0.65	1.08	0.86	1.14	1.23	1.19	1.45
1994	0.26	0.49	0.52	0.61	0.87	1.36	0.69	0.81	0.80	0.87	1.08	0.83	0.90	0.98	1.30	1.71	1.52	1.24
1995	0.23	0.35	0.42	0.59	0.90	1.22	0.57	0.60	0.72	0.84	1.01	1.00	1.41	1.09	1.19	1.24	1.32	1.31
1996	0.56	0.61	0.73	1.12	1.75	2.09	1.21	1.27	1.39	1.55	2.09	1.98	2.50	2.54	2.05	1.88	2.18	1.96

Table 6

Absolute Pricing Errors (in dollars): Black-Scholes Model

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.57	0.58	0.55	0.73	0.89	1.11	1.44	0.96	0.96	1.25	1.41	2.62	2.93	1.69	1.33	1.73	2.16	4.35
1989	0.53	0.60	0.61	0.79	1.06	1.13	1.37	0.93	0.87	1.10	1.50	3.08	3.79	1.14	1.08	1.71	2.78	8.21
1990	0.53	0.57	0.66	0.88	1.12	1.17	1.51	1.04	1.13	1.38	1.85	2.98	3.19	1.68	1.33	2.02	2.55	5.66
1991	0.49	0.51	0.59	0.71	0.96	1.26	1.31	0.98	0.98	1.22	1.76	2.67	2.79	1.56	1.46	2.11	3.38	5.12
1992	0.46	0.43	0.47	0.65	0.87	2.01	1.29	0.85	0.80	1.10	1.42	2.07	2.62	1.34	1.50	1.82	2.49	5.67
1993	0.95	0.42	0.49	0.60	0.78	1.69	2.87	0.98	0.74	1.04	1.45	1.88	5.00	1.88	1.26	1.71	3.04	8.23
1994	0.23	0.22	0.39	0.63	0.95	1.62	0.43	0.49	0.80	1.12	1.56	2.47	1.75	1.80	2.07	2.30	2.80	4.51
1995	0.28	0.23	0.39	0.69	1.05	1.78	0.66	0.43	0.74	1.17	1.87	2.65	1.75	1.15	1.33	2.09	3.37	5.37
1996	0.56	0.46	0.77	1.10	1.67	2.29	1.04	1.01	1.51	1.99	2.84	3.83	2.89	2.10	2.64	3.44	5.46	9.30

Table 7

Absolute Hedging Errors (in dollars): Generalized Black-Scholes Model

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.35	0.55	0.56	0.75	0.96	1.34	0.49	0.76	0.89	1.06	1.20	1.77	0.51	0.64	0.74	0.90	0.75	0.96
1989	0.24	0.45	0.57	0.73	0.84	0.59	0.43	0.49	0.82	0.96	1.12	1.38	0.27	0.68	0.70	0.99	2.11	1.33
1990	0.31	0.53	0.67	0.88	1.10	1.20	0.59	0.73	0.90	1.04	1.36	1.34	0.68	0.67	0.77	1.40	1.41	1.75
1991	0.19	0.48	0.60	0.79	1.08	1.01	0.45	0.66	0.89	0.96	1.36	1.08	0.52	0.78	1.08	1.93	2.07	1.47
1992	0.01	0.36	0.50	0.69	0.95	0.82	0.40	0.57	0.71	1.07	1.15	1.09	0.65	0.58	1.03	1.14	1.04	1.10
1993	0.01	0.19	0.49	0.65	0.86	1.02	0.22	0.48	0.70	0.89	1.13	1.46	0.52	0.59	0.97	0.97	1.63	1.20
1994	0.26	0.21	0.33	0.60	1.03	1.84	0.24	0.36	0.61	0.91	1.31	1.52	0.46	0.53	0.73	1.14	1.81	1.35
1995	0.13	0.22	0.35	0.64	1.00	1.13	0.20	0.34	0.67	0.97	1.35	1.22	0.35	0.56	0.89	0.99	1.43	1.14
1996	0.36	0.42	0.70	1.14	1.86	2.19	0.48	0.79	1.29	1.72	2.14	2.04	0.98	1.19	1.93	1.97	2.63	1.70

Table 8

Absolute Hedging Errors (in dollars): Hull-White Model

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.35	0.54	0.55	0.73	0.93	1.33	0.50	0.75	0.88	1.03	1.18	1.74	0.51	0.64	0.71	0.88	0.73	0.97
1989	0.24	0.45	0.55	0.69	0.80	0.58	0.45	0.49	0.80	0.94	1.10	1.36	0.27	0.68	0.69	0.97	2.04	1.33
1990	0.31	0.51	0.62	0.82	1.06	1.19	0.62	0.73	0.87	1.01	1.32	1.29	0.70	0.67	0.72	1.34	1.40	1.67
1991	0.20	0.48	0.58	0.76	1.05	1.00	0.47	0.65	0.87	0.93	1.34	1.05	0.52	0.78	1.05	1.89	2.05	1.45
1992	0.01	0.36	0.49	0.68	0.93	0.81	0.40	0.57	0.70	1.05	1.15	1.08	0.65	0.58	1.03	1.12	1.03	1.08
1993	0.01	0.19	0.48	0.64	0.86	1.01	0.25	0.48	0.69	0.88	1.12	1.42	0.54	0.60	0.96	0.95	1.60	1.20
1994	0.28	0.21	0.31	0.57	1.01	1.83	0.26	0.36	0.60	0.89	1.28	1.50	0.47	0.53	0.72	1.12	1.79	1.31
1995	0.13	0.23	0.34	0.63	0.99	1.13	0.22	0.35	0.66	0.95	1.33	1.22	0.37	0.56	0.89	0.97	1.41	1.12
1996	0.37	0.41	0.66	1.08	1.82	2.19	0.51	0.79	1.25	1.66	2.09	2.02	1.01	1.19	1.89	1.91	2.60	1.65

Table 9

Absolute Hedging Errors (in dollars): Black-Scholes Model

Year	Days to Expiration ≤ 60						$60 <$ Days to Expiration ≤ 180						$180 <$ Days to Expiration					
	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]	[1]	[2]	[3]	[4]	[5]	[6]
1988	0.35	0.54	0.55	0.74	0.95	1.35	0.51	0.71	0.88	1.04	1.21	1.76	0.54	0.58	0.72	0.89	0.75	1.03
1989	0.26	0.44	0.56	0.69	0.80	0.57	0.45	0.48	0.81	0.93	1.08	1.33	0.30	0.68	0.71	0.97	2.02	1.32
1990	0.26	0.50	0.62	0.85	1.07	1.20	0.59	0.74	0.87	1.02	1.33	1.29	0.67	0.65	0.70	1.34	1.41	1.67
1991	0.22	0.53	0.60	0.77	1.08	1.02	0.50	0.67	0.86	0.96	1.35	1.03	0.56	0.78	1.08	1.89	2.07	1.44
1992	0.01	0.37	0.50	0.68	0.94	0.82	0.41	0.59	0.71	1.05	1.14	1.10	0.64	0.64	1.02	1.12	1.05	1.09
1993	0.10	0.25	0.49	0.64	0.87	1.01	0.35	0.55	0.72	0.87	1.09	1.27	0.61	0.65	1.00	0.94	1.59	1.22
1994	0.63	0.27	0.32	0.58	1.03	1.85	0.36	0.39	0.61	0.89	1.25	1.42	0.49	0.54	0.73	1.15	1.81	1.33
1995	0.13	0.36	0.39	0.65	1.00	1.18	0.46	0.49	0.70	0.94	1.31	1.25	0.52	0.68	0.96	0.97	1.39	1.04
1996	0.47	0.53	0.71	1.10	1.87	2.22	0.63	0.90	1.29	1.66	2.07	2.05	1.16	1.24	1.90	1.91	2.59	1.69