

# Policy choice and partial identification\*

## Preliminary and incomplete draft

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### Abstract

Many methodological debates in microeconometrics are driven by the tension between “what we can get” (identification) and “what we want” (parameters of interest). This paper proposes to consider models of policy choice which allow for a joint formal discussion of both issues. In particular, we study the problem of optimal treatment assignment based on partial identification of conditional average treatment effects. Partial identification of conditional average treatment effects maps into a partial ordering of policies in terms of social welfare, and into partial identification of the set of optimal policies among a given set of feasible policies.

This paper gives geometric characterizations of the identified partial ordering of policies, and derives conditions for restricted policy sets to be completely ordered or completely unordered. Such conditions map sets of feasible policies into requirements on data that allow to rank these policies. Generalizing to non-linear objective functions, it is then shown that policy effects are partially identified if and only if the policy objective is a robust statistic in the sense of having a bounded influence function. The paper finally characterizes maximum regret and Bayesian expected regret under partial identification, relative to point identification. This allows to provide a decision-theoretic comparison of identification approaches.

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## 1 Introduction

In recent years, microeconometrics has seen lively methodological debates between proponents of “causal” and those of “structural” approaches, see for instance Deaton (2009), Im-

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bens (2010), Angrist and Pischke (2010), and Nevo and Whinston (2010). In these debates, proponents of “causal” approaches have emphasized the necessity of credible identification. This entails recognition of the limits of what we can get from given data. Proponents of “structural” approaches have emphasized the importance of estimating parameters which allow to evaluate counterfactual policies. Structural parameters are those that are invariant under relevant counterfactual scenarios.

The present paper builds on these arguments and proposes a framework that allows to formally discuss both identification and evaluation of counterfactual policies. The policies  $h$  considered allocate a binary treatment  $D$  based on covariates  $X$ ,  $h(X) = P(D = 1|X)$ . The set of feasible policies is possibly subject to constraints. We are interested in identification approaches that partially identify conditional average treatment effects  $g(X) = E[Y^1 - Y^0|X]$ . The planner’s objective function takes the form  $SWF = E[Y]$ . A policy  $h^a$  is therefore preferred to a policy  $h^b$  if  $E[(h^a(X) - h^b(X))g(X)] > 0$ , as will be discussed below. In this setup, it is possible to establish a partial preference ranking<sup>1</sup> of policies  $h$  under weaker (and therefore more credible) assumptions than would be needed to point-identify the underlying structural relationship  $g$ . Similarly, it is possible to determine “near-optimal” policies without point identification of treatment effects. This paper is interested in the question of identification of the policy ranking, and of optimal policies.<sup>2</sup> In particular, this paper discusses the relationship between policy constraints, objective functions, and data requirements, which are such that it is possible to rank the feasible policy alternatives.

Section 2 provides geometric characterizations of the partial preference ranking of policies  $h$ , in terms of social welfare  $SWF$ , that can be derived from an identified set for conditional average treatment effects  $g$ . In particular, it is shown that two policies  $h^a$  and  $h^b$  can be ranked if the policy difference  $h^{ab} = h^a - h^b$  lies in the dual cone or the polar cone of the identified set for  $g$ ;  $h^a$  and  $h^b$  can not be ranked if  $h^{ab}$  is orthogonal to some  $g$  in the identified set. Modifying results from the axiomatic decision theory literature (in particular Bewley (2002)), we then show that the identified preference order satisfies a property called independence, and any partial order satisfying independence is “as if” it did arise from some identified set for  $g$ . Section 2 also gives necessary and sufficient conditions for sets of feasible policies which are subject to linear budget constraints to be completely ordered, and for them to be completely unordered. It is shown that such sets of feasible policies can only be completely ordered if they are one-dimensional. Such sets are completely unordered if and only if we cannot preclude the possibility that treatment effects are proportional to the covariate specific cost of treatment.

Section 3 generalizes the setup introduced in section 1.1, and discussed in section 2, by

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<sup>1</sup>What is called the identified partial ranking in this paper has been called dominance, Pareto criterion, or Knightian preferences in different contexts; see for instance Manski (2011), Stoye (2011b), and Bewley (2002).

<sup>2</sup>This stands in contrast to papers such as Manski (2011), who discusses the problem of policy choice given an identified set for policy effects. Prescriptions in the literature for this latter problem include Bayesian expected welfare maximization, maximin, and minimax regret.

allowing for nonlinear objective functions  $\phi$  which are functionals of the outcome distribution  $F(Y)$ . Section 3 focuses on local policy changes relative to a status quo policy  $h^0$ . For such local policies, social welfare  $\phi(F)$  can be approximated, for the purpose of welfare rankings, using the functional derivative  $\partial\phi/\partial F$ , which effectively linearizes the problem again. We find that the effect of local policy changes is partially identified, with finite bounds, if and only if  $\partial\phi/\partial F$  is bounded on the set of feasible counterfactual outcome distributions. This is equivalent to  $\phi$  having a bounded influence function, which is a central concept in robust statistics. Section 4 returns to the basic linear setup. It characterizes maximal and expected Bayesian welfare as a function of the identified set for  $g$ . This allows to formally compare different identification approaches based on the social welfare they allow to attain.

Several strands of literature have been important in developing the ideas for this paper. The problem of optimal treatment assignment based on covariates has been analysed by Manski (2004), Dehejia (2005), Bhattacharya and Dupas (2008), Hirano and Porter (2009), and Stoye (2011a) among others. These papers focus on the decision-theoretic properties of policy choice based on finite samples from distributions point-identifying treatment effects. Their results apply in particular to (ideal) randomized experiments. In contrast, the present paper is interested in policy choice based on knowledge of distributions which only partially identify treatment effects. This is of relevance in particular in the context of observational studies in the absence of functional form assumptions. For analytical clarity, this paper sidesteps the additional complications following from sampling uncertainty.

Manski (2011) studies a setup very similar to the one in the present paper, focusing on the question of how to choose a treatment assignment policy, given a partially identified set for treatment effects. The present paper poses different questions, asking what a given identified set allows us to learn about policy rankings, and how to compare different identification approaches in terms of the welfare attainable if policy is chosen based on them. Hansen and Sargent (2008) also study, in a rather different context, questions related to the one of Manski (2011); how to choose policy to be robust under a set of model alternatives.

The decision theory literature since Knight (1921) has a long tradition studying choice in the absence of subjective probability distributions, i.e., under ambiguity. Formally related to the present paper are in particular Bewley (2002) and Ryan (2009), building on the classic Anscombe and Aumann (1963). In economic decision theory, ambiguity is primarily an ex-post characterization of behaviour satisfying certain axioms. In contrast, in the present context it arises naturally from econometric models in the absence of functional form assumptions. Both stand in contrast to the issue of ambiguity in frequentist statistical decision theory, which does not impose prior distributions on model parameter values; see Stoye (2011b) and the discussion in section 2.2.

Partial identification of treatment effects has been analyzed in the pioneering work of

Manski. The results in this paper apply to any approach yielding partial identification of treatment effects. Two identification approaches are discussed in more detail in Appendix A, nonparametric instrumental variables (see Manski (2003)) and panel data with an assumption about marginal stationarity of unobserved heterogeneity (see Chernozhukov et al. (2010)).

The relationship between relevant policy sets and parameters of interest has been considered recently by Chetty (2009) and Graham et al. (2008). Like the present paper, both of these make arguments implying that evaluation of the relative merits of different policies only requires knowledge of some parameters, not full knowledge of the underlying structural relationships. Sen (1995) also raises a related point, arguing that even if there is disagreement over the exact trade-offs between policy objectives, social consensus might be achieved over the ranking of policy alternatives.

The rest of the paper is structured as follows. Subsection 1.1 formally introduces the setup discussed in this paper. In section 2, the properties of the partial preference ranking on the set of feasible policies induced by partially identified treatment effects are analysed and geometrically interpreted. In section 3, the results are generalized to nonlinear objective functions. In section 4, identification of optimal policies is studied, and several characterizations of the welfare (regret) attainable given an identification approach are provided. In section 5, the results are applied to the evaluation of the impact of segregation, relating the present paper to Graham et al. (2008). Section 6 concludes. Appendix A reviews partial identification of treatment effects based on instrumental variables and panel data. Appendix B provides Bayesian priors for IV and panel data models. In Appendix C, the relationship between (distributional) preferences of the planner and the identified partial order are discussed for a particular family of preferences. All proofs are relegated to Appendix D.

## 1.1 Setup

Suppose the outcome of interest,  $Y$ , is generated by a structural relationship of the form

$$Y = f(X, D, \epsilon), \tag{1}$$

where the treatment  $D$  lies in  $\{0, 1\}$  and  $Y \in [0, 1]$ . Denote the potential outcomes by  $Y^d = f(X, d, \epsilon)$  for  $d = 0, 1$ . Consider treatment assignment policies where units characterized by  $X$  are randomly assigned to treatment  $D = 1$  with probability  $h(X)$ :

$$D|X, \epsilon \sim \text{Ber}(h(X)). \tag{2}$$

This includes as a special case deterministic policies of the form  $h(X) \in \{0, 1\}$ , where  $D = h(X)$ . Suppose finally that the policy objective function takes the form

$$SWF = E[Y]. \tag{3}$$

The results of this paper immediately generalize to the case of bounded interval support of  $Y$ . The assumption of binary treatment  $D$  is important, although the identification results discussed in appendix A.1 and A.2 do generalize to the case of larger, finite support of  $D$ . The size of the identified set is increasing in the size of the support of  $D$ , however, and identification disappears for continuously distributed  $D$ .

The form of the objective function  $SWF$  is restrictive in that it imposes additive separability of social welfare across units of observation. It is however consistent with various preferences over inequality or risk, group specific costs of treatment, etc., if we interpret  $Y$  to be a possibly nonlinear transformation of measurable outcomes  $Y^{obs}$ , as well as  $X$  and  $D$ :  $Y = Y(Y^{obs}, X, D)$ . Section 3 will generalize this setup by allowing for objective functions of the form  $SWF = \phi(F)$ , where  $\phi$  is a (smooth) functional of the outcome distribution  $F(Y)$ .

There are many setups of interest that can be subsumed under this framework, where  $g$  might reflect a technological or a behavioral relationship. Decision problems that fit into this framework include (i) the assignment of income support programs by a policymaker interested in labor market outcomes, as in Dehejia (2005); (ii) the allocation of indivisible capital goods to units of production by profit maximizing firm owners; (iii) the assignment of a medical treatment by doctors maximizing some health outcomes of their patients; (iv) the decision of whether or not to attend college by students taking into account the economic returns (among other factors), as in Card (2001); (v) the assignment of students to integrated or segregated classes by an educational policymaker interested in maximizing average rescaled test-scores, as in Graham et al. (2008). Some of these applications will be discussed in more detail in section 5. In all of these applications, we might expect the available data to only partially identify average treatment effects.

An important limitation of the setup considered here is that it does not allow for incentive compatibility constraints on feasible policy sets. Incentive compatibility introduces a particular form of non-separability of constraints across treatment units which complicates the analysis, yet is central for many interesting optimal policy problems in mechanism design and optimal taxation, such as those discussed in Chetty (2009). Optimal policy under partial identification and informational constraints will be analyzed in a separate paper.

## 2 Welfare ranking of policies

Partial identification of the potential outcome distributions  $P(Y^d|X)$  for  $d = 0, 1$  implies a partial ordering of the set of policies  $h$ . Consider a policy  $h^a$ , where  $D^a|X, \epsilon \sim Ber(h^a(X))$ . Define  $Y^a := f(X, D^a, \epsilon)$  and  $SWF^a := E[Y^a]$ . Define  $Y^b, SWF^b$  and  $D^b$  similarly for a policy  $h^b$ . Denote the difference of the probability of assignment to treatment between these two policies by

$$h^{ab} := h^a - h^b \tag{4}$$

and let the difference in social welfare achieved by the two policies be given by

$$SWF^{ab} := SWF^a - SWF^b = E[Y^a] - E[Y^b]. \tag{5}$$

The social welfare ordering between two policies  $h^a, h^b$  is identified if and only if

$$\text{sign}(SWF^{ab}) = \text{sign}(SWF^a - SWF^b) = \text{sign}(E[Y^a] - E[Y^b])$$

is identified. Note that

$$\begin{aligned} SWF^{ab} &= E[Y^a - Y^b] = E[(D^a - D^b)(Y^1 - Y^0)] \\ &= E[(h^a(X) - h^b(X))(Y^1 - Y^0)] \\ &= E[h^{ab}(X)g(X)], \end{aligned} \tag{6}$$

where  $g(X)$  is the conditional average treatment effect defined as

$$g(X) := E[Y^1 - Y^0|X]. \tag{7}$$

This holds because of the conditional independence between  $Y^1, Y^2$  and  $D^a, D^b$  given  $X$ . This paper will use  $g$  throughout to denote such conditional average treatment effects.

Suppose we have (tight) upper and lower bounds,  $\overline{Y^d}(X)$  and  $\underline{Y^d}(X)$  on the conditional average structural function  $E[Y^d|X]$  for  $d = 0, 1$ . Suppose furthermore that there are no (data-implied) restrictions across  $X$ , nor across treatments  $d$ , on the conditional average structural function. Appendix A.1 and A.2 discuss assumptions that give rise to such bounds, building on Manski (2003) and Chernozhukov et al. (2010).<sup>3</sup> These bounds imply corresponding

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<sup>3</sup>It is recommended to read appendix A.1 and A.2 before moving on to section 2.1. This will put the results in a more specific context.

bounds on the average treatment effects,

$$\begin{aligned}\bar{g}(X) &= \overline{Y^1}(X) - \underline{Y^0}(X) \\ \underline{g}(X) &= \underline{Y^1}(X) - \overline{Y^0}(X),\end{aligned}\tag{8}$$

so that the identified set for  $g$  is given by

$$\mathcal{G} = \{g : \underline{g}(x) \leq g(x) \leq \bar{g}(x) \forall x\}.\tag{9}$$

Note that the worst case scenario, from a planner's perspective, is given by  $(Y^0, Y^1) = (\underline{Y^0}, \underline{Y^1})$ , independently of the chosen policy. The corresponding conditional average treatment effect is given by  $g^{mm} := \underline{Y^1} - \underline{Y^0}$ . This will be important in the context of maximin policy choice (explaining the superscript  $mm$  for maximin), as discussed in section 4 .

The identified set for  $SWF^{ab}$  is given by

$$SWF^{ab} \in [\underline{SWF^{ab}}, \overline{SWF^{ab}}],$$

where

$$\begin{aligned}\overline{SWF^{ab}} &= \max_{g \in \mathcal{G}} E[h^{ab}(X)g(X)] = E[h^{ab}(X)g^u(X)] \\ \underline{SWF^{ab}} &= \min_{g \in \mathcal{G}} E[h^{ab}(X)g(X)] = E[h^{ab}(X)g^l(X)],\end{aligned}\tag{10}$$

and

$$\begin{aligned}g^u(X) &= \operatorname{argmax}_{g \in \mathcal{G}} E[h^{ab}(X)g(X)] = \bar{g}(X)\mathbf{1}(h^{ab}(X) \geq 0) + \underline{g}(X)\mathbf{1}(h^{ab}(X) < 0) \\ g^l(X) &= \operatorname{argmin}_{g \in \mathcal{G}} E[h^{ab}(X)g(X)] = \bar{g}(X)\mathbf{1}(h^{ab}(X) < 0) + \underline{g}(X)\mathbf{1}(h^{ab}(X) \geq 0).\end{aligned}\tag{11}$$

Two policies  $h^a, h^b$  are strictly ordered if and only if

$$\operatorname{sign}(\overline{SWF^{ab}}) = \operatorname{sign}(\underline{SWF^{ab}}) \neq 0,\tag{12}$$

i.e., if either

$$\underline{SWF^{ab}} > 0$$

or

$$\overline{SWF^{ab}} < 0.$$

As an aside, consider the special case of deterministic  $h^a, h^b$ . In this case, the policy difference  $h^{ab}$  only takes on the values  $\{-1, 0, 1\}$ , so that we can define

$$\begin{aligned}\mathcal{X}^a &= \{X : h^a(X) - h^b(X) = 1\} \\ \mathcal{X}^b &= \{X : h^a(X) - h^b(X) = -1\}.\end{aligned}\tag{13}$$

Using this notation, we can rewrite

$$\begin{aligned}SWF^{ab} &= E[g(X)h^{ab}(X)] \\ &= E[g(X)|X \in \mathcal{X}^a]P(\mathcal{X}^a) - E[g(X)|X \in \mathcal{X}^b]P(\mathcal{X}^b),\end{aligned}\tag{14}$$

and therefore

$$\begin{aligned}\overline{SWF}^{ab} &= E[\bar{g}(X)|X \in \mathcal{X}^a]P(\mathcal{X}^a) - E[\bar{g}(X)|X \in \mathcal{X}^b]P(\mathcal{X}^b) \\ \underline{SWF}^{ab} &= E[\underline{g}(X)|X \in \mathcal{X}^a]P(\mathcal{X}^a) - E[\underline{g}(X)|X \in \mathcal{X}^b]P(\mathcal{X}^b).\end{aligned}\tag{15}$$

## 2.1 The geometry of the set of ordered policies

Throughout this paper, we will consider the space of bounded measurable functions of  $X$ , equipped with the inner product

$$\langle h, g \rangle := E[h(X)g(X)].\tag{16}$$

and with the norm  $\|g\| = \sqrt{\langle g, g \rangle} = \sqrt{E[g^2(X)]}$ . In this notation, we can write

$$SWF^{ab} = E[Y^a - Y^b] = E[(h^a(X) - h^b(X))g(X)] = \langle h^{ab}, g \rangle.\tag{17}$$

Define the set of policies

$$\mathcal{H} = \{h(\cdot) : 0 \leq h(X) \leq 1\}.\tag{18}$$

The corresponding set of policy differences,

$$d\mathcal{H} = \mathcal{H} - \mathcal{H} = \{h^{ab} = h^a - h^b : h^a, h^b \in \mathcal{H}\} = \{h : \sup(|h|) \leq 1\},\tag{19}$$

is the unit ball of  $X$ -measurable functions with respect to the sup norm. Suppose that the identified set for the conditional average treatment effect  $g$  is given by  $\mathcal{G}$ . This includes as a special case rectangular sets  $\mathcal{G}$ , as they arise under nonparametric instrumental variables or panel data assumptions, see appendix A.1 and A.2:

$$\mathcal{G} = \{g(\cdot) : g(X) \in [\underline{g}(X), \bar{g}(X)]\}.\tag{20}$$

We allow for both finite and infinite support of  $X$ .

If conditional average treatment effects are given by  $g(X)$ , this gives rise to the following social welfare ranking between policies

$$\begin{aligned} h^a \succ^g h^b &: \Leftrightarrow \langle h^{ab}, g \rangle > 0 \\ h^a \succeq^g h^b &: \Leftrightarrow \langle h^{ab}, g \rangle \geq 0. \end{aligned} \quad (21)$$

Equation (21) defines a complete order<sup>4</sup> of all policy pairs  $h^a, h^b$ . If  $g$  is known to lie in the identified set  $\mathcal{G}$ , we can define the identified welfare ranking

$$\begin{aligned} h^a \succ^{\mathcal{G}} h^b &: \Leftrightarrow \langle h^{ab}, g \rangle > 0 \forall g \in \mathcal{G} \\ h^a \succeq^{\mathcal{G}} h^b &: \Leftrightarrow \langle h^{ab}, g \rangle \geq 0 \forall g \in \mathcal{G}. \end{aligned} \quad (22)$$

The relation  $\succeq^{\mathcal{G}}$  is a partial order of all policy pairs  $h^a, h^b$ . We have

$$g \in \mathcal{G} \Rightarrow (h^a \succeq^{\mathcal{G}} h^b \Rightarrow h^a \succeq^g h^b), \quad (23)$$

similarly for  $\succ^g, \succ^{\mathcal{G}}$ . We can not necessarily order all policy pairs based on knowledge of  $\mathcal{G}$ . If, however,  $h^a \succeq^{\mathcal{G}} h^b$ , then we know that  $h^a$  is weakly preferred to  $h^b$ ,  $h^a \succeq^g h^b$ , even though we have only partially identified the structural relationship  $g$ .

The relationship between the set  $\mathcal{G}$  and the preference ordering  $\succeq^{\mathcal{G}}$  can be given a useful geometric interpretation which will be discussed in this section. The dual cone of a set  $\mathcal{G}$  is defined as the set

$$\widehat{\mathcal{G}} = \{h : \min_{g \in \mathcal{G}} \langle h, g \rangle \geq 0\} = \{h : \langle h, \mathcal{G} \rangle \subset [0, \infty)\}. \quad (24)$$

This is the set of all  $h$  such that the angle between  $h$  and  $g$  is no more than  $90^\circ$  for all  $g$  in  $\mathcal{G}$ . Equivalently, it is the intersection over all  $g \in \mathcal{G}$  of the halfspaces  $\{h : \langle h, g \rangle \geq 0\}$ . The polar cone is given by

$$\mathcal{G}^* = -\widehat{\mathcal{G}} = \{h : \max_{g \in \mathcal{G}} \langle h, g \rangle \leq 0\} = \{h : \langle h, \mathcal{G} \rangle \subset (-\infty, 0]\}. \quad (25)$$

This is the set of all  $h$  such that the angle between  $h$  and  $g$  is at least  $90^\circ$  for all  $g$  in  $\mathcal{G}$ , see

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<sup>4</sup>A partial order satisfies transitivity,  $h^a \succeq h^b$  and  $h^b \succeq h^c$  implies  $h^a \succeq h^c$ . A complete order satisfies additionally that for all  $h^a, h^b$ , either  $h^a \succeq h^b$  or  $h^b \succeq h^a$ ; see Mas-Colell et al. (1995).

for instance Bertsekas et al. (2003). We have

$$h^a \succeq^C h^b :\Leftrightarrow h^{ab} \in \widehat{\mathcal{G}} \quad (26)$$

$$h^b \succeq^{\mathcal{G}} h^a :\Leftrightarrow h^{ab} \in \mathcal{G}^*. \quad (27)$$

Let  $\overline{\mathcal{K}}$  denote the closure of a set  $\mathcal{K}$  with respect to the norm  $\|\cdot\|$ , and  $\mathcal{K}^\circ$  the interior of  $\mathcal{K}$ .

**Lemma 1 (The maximal set of ordered policy pairs)** *Suppose that the identified set  $\mathcal{G}$  is convex,  $0 \notin \overline{\mathcal{G}}$  and  $\operatorname{argmin}_{g \in \overline{\mathcal{G}}} \|g\|$  exists.*

*Then  $\mathcal{G}$  is uninformative about the ordering of  $h^a, h^b$  (neither  $h^a \succeq^{\mathcal{G}} h^b$  nor  $h^b \succeq^{\mathcal{G}} h^a$ ) if and only if*

$$h^{ab} \in d\mathcal{H} \setminus \left( \widehat{\mathcal{G}} \cup \mathcal{G}^* \right) = d\mathcal{H} \cap \left( \bigcup_{g \in \mathcal{G}} g^\perp \right)^\circ, \quad (28)$$

where  $g^\perp = \{h : \langle h, g \rangle = 0\}$  is the orthocomplement of  $g$ .

The proof of this lemma, and all further proofs, can be found in appendix C. The equality in equation (28) requires the existence of a separating hyperplane between 0 and  $\mathcal{G}$ , which follows from the existence of  $\operatorname{argmin}_{g \in \overline{\mathcal{G}}} \|g\|$ . This always holds for rectangular  $\mathcal{G}$ , as in equation (20), and if  $X$  has finite support. For rectangular  $\mathcal{G}$ , an element of  $\operatorname{argmin}_{g \in \overline{\mathcal{G}}} \|g\|$  is given by

$$h' = \begin{cases} \frac{g}{\bar{g}} & \text{if } |g| < |\bar{g}| \\ \bar{g} & \text{else.} \end{cases}$$

Existence of such a separating hyperplane more generally follows from the Hahn-Banach theorem if  $\mathcal{G}^\circ$  is not empty.

Define  $d\mathcal{H}^{\max}$  to be the maximal set of policy differences  $h^{ab}$  such that  $h^a, h^b$  are (weakly) ordered by  $\succeq^{\mathcal{G}}$ ,

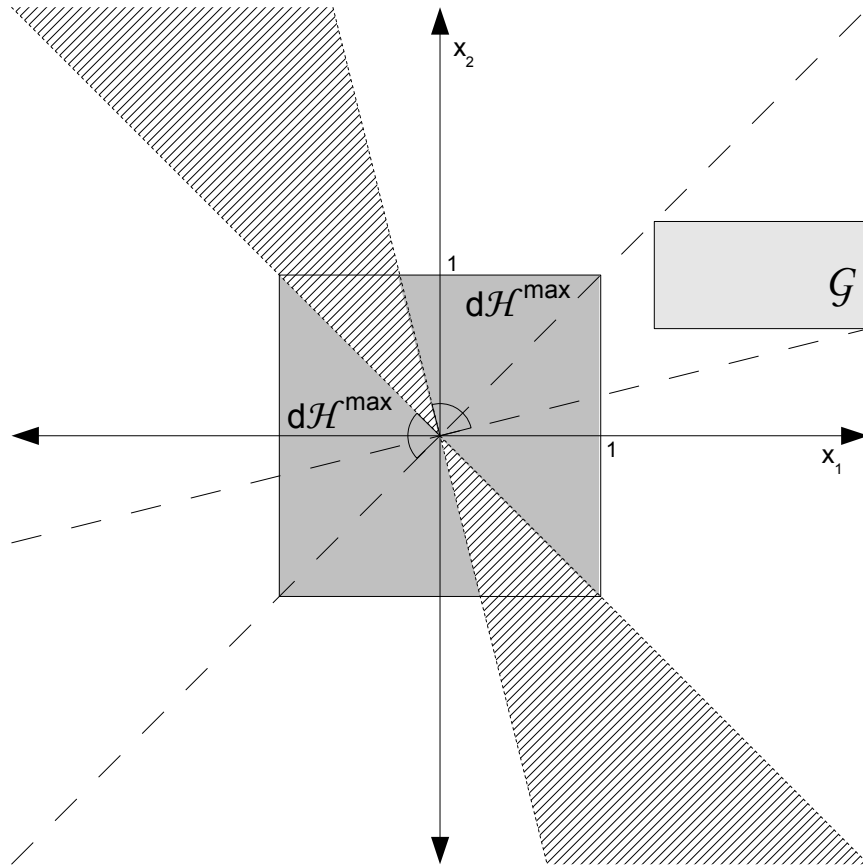
$$d\mathcal{H}^{\max} = d\mathcal{H} \cap \left( \widehat{\mathcal{G}} \cup \mathcal{G}^* \right).$$

The set of corresponding policy pairs  $h^a, h^b$  is given by

$$\left\{ (h^a, h^b) \in \mathcal{H} \times \mathcal{H} : h^a - h^b \in d\mathcal{H}^{\max} \right\}. \quad (29)$$

Figure 1 illustrates lemma 1 for the case where  $X$  has two points of support. The identified region  $\mathcal{G}$  for  $g(\cdot)$  is rectangular, since we have assumed separate bounds on  $g(X)$  for each  $X$ . The maximal set of policy differences corresponding to weakly ordered policy pairs,  $d\mathcal{H}^{\max}$ , is given by the intersection of the set of feasible policy differences (the square in the center), with the dual cone and the polar cone of  $\mathcal{G}$ . The dual cone is the set in the upper right part

Figure 1: IDENTIFICATION REGION OF THE CONDITIONAL AVERAGE TREATMENT EFFECT  $g$  AND THE SET OF ORDERED POLICIES IF  $\text{supp}(X) = \{x_1, x_2\}$



*Notes:* This figure illustrates the geometry of the relationship between the identification region  $\mathcal{G}$  for  $g$ , and the set  $d\mathcal{H}^{\max}$  of policy-differences  $h^{ab}$  such that  $h^a, h^b$  are (weakly) ordered by  $\succeq^{\mathcal{G}}$ . See text for a discussion.

of the figure, it contains all vectors that have an angle of  $90^\circ$  or less with all vectors in  $\mathcal{G}$ . The polar cone is the set in the lower left part of the figure, it contains all vectors that have an angle of  $90^\circ$  or more with all vectors in  $\mathcal{G}$ . The set of policy differences which can not be strictly ranked is the set of vectors which are orthogonal to some element of  $\mathcal{G}$ .

## 2.2 Characterization of implied preferences and the literature on axiomatic decision theory

At this point it is useful to take a step back and compare the characterization of the (incomplete) preference order  $\succ^{\mathcal{G}}$  obtained in the last subsection to the decision theoretic literature on Knightian preferences, in particular Bewley (2002) and the review in Ryan (2009). There are a number of formal parallels, as well as differences in terms of interpretation, between the setup considered here and the literature on axiomatic decision theory.

The first difference concerns the space over which preferences are defined. The positive, axiomatic decision theory literature following Anscombe and Aumann (1963) considers preferences over acts. Acts are lotteries with known outcomes, known (“objective”) probability distributions conditional on the states of the world, and unknown probability distributions across states of the world. The present setup, in contrast, considers preferences over policies. The policies considered (randomly) assign treatment as a function of covariates, with defined probabilities conditional on covariates, a known distribution of covariates, and unknown (but bounded, as a function of the data) conditional average outcomes.

The second difference concerns the question of interest. The positive decision theory literature seeks to impose plausible restrictions on actual human behaviour, based on experimental and other evidence. From these restrictions characterizations of preferences and behaviour are then derived; in the case of Bewley (2002) in terms of a set of priors over states of the world and an inertia assumption on behaviour. In the “normative” setup of the present paper, the reverse question is discussed. We start with an identified set of conditional average treatment effects functions (the formal equivalent of the set of priors in Bewley (2002)), and then derive preferences and behaviour.

Having pointed out these differences, the rest of this subsection discusses to what extent the results of Bewley (2002) can be modified to apply in the present context. This section will follow the exposition in Ryan (2009), with the necessary modifications. The first thing to note is that the relationship  $\succ^{\mathcal{G}}$  satisfies the property called independence in the decision theory literature:

**Definition 1 (Independence)** *The relationship  $\succ$  satisfies independence if, for all  $h^a, h^b, h^c \in \mathcal{H}$ , and all  $\alpha \in (0, 1)$ , we have that  $h^a \succ h^b$  if and only if*

$$\alpha h^a + (1 - \alpha)h^c \succ \alpha h^b + (1 - \alpha)h^c.$$

That  $\succ^{\mathcal{G}}$  satisfies independence is immediate from definition (22). Remarkably, as we shall prove now, the reverse also holds: Any partial order  $\succ$  which satisfies independence can be represented as  $\succ^{\mathcal{G}}$  for some convex set  $\mathcal{G}$ . The following lemma helps to prove this. It is an adaptation of lemma 2 and corollary 2 from Ryan (2009).

**Lemma 2** *Suppose the relationship  $\succ$  on  $\mathbb{R}^{\mathcal{X}}$  satisfies independence. Then, for all  $h^a, h^b, h^c \in \mathbb{R}^{\mathcal{X}}$ , and all  $\alpha > 0$ ,*

1.  $h^a \succ h^b$  iff  $\alpha h^a \succ \alpha h^b$ ,
2.  $h^a \succ h^b$  iff  $h^a + h^c \succ h^b + h^c$ ,
3.  $\{h : h \succ h^a\} = \{h : h \succ 0\} + h^a$ ,
4.  $\{h : h \succ 0\}$  is a convex cone.

Using our previous notation, items 3 and 4 of lemma 2 show that, given independence,  $h^a \succ h^b$  iff  $h^{ab} = h^a - h^b$  lies in the convex cone  $\tilde{\mathcal{G}} := \{h : h \succ 0\}$ . Define the set  $\mathcal{G}$  as the dual cone of  $\tilde{\mathcal{G}}$ .<sup>5</sup> By equation (27),  $h^a \succ^{\mathcal{G}} h^b$  iff  $h^{ab} = h^a - h^b$  lies in the dual cone  $\hat{\mathcal{G}}$  of  $\mathcal{G}$ . The claim now follows if we can show that  $\hat{\mathcal{G}}$  equals  $\tilde{\mathcal{G}}$ . But this follows from the dual cone theorem of convex analysis, which asserts that the dual cone  $\hat{\mathcal{G}}$  of the dual cone  $\mathcal{G}$  of a convex cone  $\tilde{\mathcal{G}}$  equals the initial convex cone  $\tilde{\mathcal{G}}$ ; see for instance Bertsekas et al. (2003). We have thus proven the following proposition, which modifies the results discussed in section 4 of Ryan (2009).

**Proposition 1** *A partial order  $\succ$  on  $\mathbb{R}^{\mathcal{X}}$  satisfies independence if and only if it can be represented as  $\succ^{\mathcal{G}}$  for some convex set  $\mathcal{G}$ .*

Proposition 1 is interesting, since it fully characterizes the range of (incomplete) preferences which can arise as a consequence of partial identification of policy effects.

This section concludes by discussing the relationship of the questions discussed here to the statistical and econometric decision theory literature; recent contributions to this literature include Hirano and Porter (2009), Manski (2004), and Stoye (2011b). We can think of frequentist statistics as being concerned with a problem of choice under ambiguity, too: The models considered by frequentists assign conditional probabilities to the data, given parameter values. In contrast to Bayesian statistics, there is no (subjective) distribution across parameters in frequentist statistics, however. The problem of choice of decision rules by frequentists is thus a problem of choice under ambiguity.

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<sup>5</sup>Equivalently, we could take any convex set  $\mathcal{G}$  such that the minimal cone containing  $\mathcal{G}$ ,  $\mathbb{R}^+ \cdot \mathcal{G}$ , equals the dual cone of  $\tilde{\mathcal{G}}$ .

There are thus three different roles for choice under ambiguity in the literature, which should not be confused: As an “as if” characterization of positive behaviour satisfying certain axioms. As the (inferential) problem of frequentist statistics. And as a consequence of partial identification of choice-relevant parameters, as in the present paper.

### 2.3 Sufficient conditions

Lemma 1 characterized the set of all policy pairs which are ordered by  $\succeq^{\mathcal{G}}$ . This section is going to give a sufficient condition for a pair of policies  $h^a, h^b$  to be ordered, i.e., for identification of  $\text{sign}(SWF^{ab})$ . Assume the identification region  $\mathcal{G}$  is rectangular, and let the function  $w$  describe the width of the identified set for  $g(X)$ ,

$$w(X) = \bar{g}(X) - \underline{g}(X). \quad (30)$$

Appendix A.1 and A.2 give expressions for  $w$  which have the interesting feature that they do not depend on the distribution of  $Y$ , but are solely a function of the strength of the instrument (in A.1), or the variability of treatment within cross-sectional units over time (in A.2).

In the notation of the introduction to section 2 and of subsection 2.1, we have

$$SWF^{ab} - \underline{SWF}^{ab} = \langle h^{ab}, g - g^l \rangle = \langle |h^{ab}|, |g - g^l| \rangle \quad (31)$$

and

$$\overline{SWF}^{ab} - SWF^{ab} = \langle h^{ab}, g^u - g \rangle = \langle |h^{ab}|, |g^u - g| \rangle. \quad (32)$$

By definition of  $w$

$$|g - g^l| \leq |g^u - g^l| = \bar{g}(X) - \underline{g}(X) = w(X), \quad (33)$$

similarly for  $|g^u - g|$ , and therefore

$$\begin{aligned} SWF^{ab} - \underline{SWF}^{ab} &\leq \langle |h^{ab}|, w \rangle \\ \overline{SWF}^{ab} - SWF^{ab} &\leq \langle |h^{ab}|, w \rangle \end{aligned} \quad (34)$$

If  $h^a, h^b$  are deterministic policies then  $|h^{ab}| \in \{0, 1\}$ , and we can rewrite the right hand side of inequality (34) as

$$\langle |h^{ab}|, w \rangle = E[\mathbf{1}(h^{ab} \neq 0) \cdot w] = E[w | \mathcal{X}^{ab}] P(\mathcal{X}^{ab}), \quad (35)$$

where  $\mathcal{X}^{ab} = \{x : h^a(x) \neq h^b(x)\}$ . Recall  $\langle h^{ab}, g \rangle = SWF^{ab}$ . We get the following lemma:

**Lemma 3 (Sufficient condition)** *Under the assumptions maintained in this section, if*

$$\langle |h^{ab}|, w \rangle = \langle h^{ab}, w \cdot \text{sign}(h^{ab}) \rangle < \left| \langle h^{ab}, g \rangle \right|, \quad (36)$$

then  $\text{sign}(SWF^{ab})$  is identified, and hence either  $h^a \succ^{\mathcal{G}} h^b$  or  $h^b \succ^{\mathcal{G}} h^a$ . If  $h^a, h^b$  are deterministic policies, this condition can be rewritten as

$$E[w|\mathcal{X}^{ab}] < |SWF^{ab}|/P(\mathcal{X}^{ab}). \quad (37)$$

Note that the left hand side of inequality (36) is the width of the identified set (interval) for  $SWF^{ab}$ , while the right hand side is equal to  $|SWF^{ab}|$ .

This lemma can also be interpreted by considering a hypothetical identified set

$$\mathcal{G}' = \{g' : g(X) - w(X) \leq g'(X) \leq g(X) + w(X)\}. \quad (38)$$

The assumptions maintained in this section imply  $\mathcal{G} \subset \mathcal{G}'$ . The set of  $h^a, h^b$  for which condition (36) holds is exactly the set of  $h^a, h^b$  for which either  $h^a \succ^{\mathcal{G}'} h^b$  or  $h^b \succ^{\mathcal{G}'} h^a$ .

## 2.4 Restricted policy sets

We might hope that some restricted sets of policies are totally ordered by  $\succeq^{\mathcal{G}}$ , even if the full set  $\mathcal{H}$  is not. A particularly interesting set of restrictions are policy sets which lie in affine subspaces intersected with  $\mathcal{H}$ . Such restrictions might arise for instance as a consequence of linear budget constraints of the form  $E[h(X)c(X)] = \langle h, c \rangle = C$ , where  $c(X)$  is the (average) cost of treatment for units characterized by  $X$ .

The following lemma characterizes when policy sets subject to affine restrictions are totally ordered.

### **Lemma 4 (Affine policy sets which are totally ordered by $\succeq^{\mathcal{G}}$ )**

*Suppose  $\mathcal{G}$  has nonempty interior  $\mathcal{G}^\circ$ .*

*Let  $\mathcal{H}'$  be a set of policies which is given by the intersection of an affine space with  $\mathcal{H}$ .*

*Then, if  $\mathcal{H}'$  is totally ordered by  $\succeq^{\mathcal{G}}$ , then  $\mathcal{H}'$  is at most one dimensional.*

An important example of such one dimensional policy spaces arises if  $h$  is restricted to be constant in  $X$ ; or, more generally, if  $h$  has to be equal to 0 on a subset of  $X$  and constant on the rest of its support. Such restrictions might arise for instance if there is a non-discrimination requirement that all individuals be treated equally or with equal probability.

Lemma 4 characterizes when sets subject to affine restrictions are totally ordered. The following lemma characterizes the opposite extreme; sets subject to linear budget constraints which can not be ordered at all.

### **Lemma 5 (Affine policy sets for which $\mathcal{G}$ is completely uninformative about $\succeq^{\mathcal{G}}$ )**

*Suppose that  $\mathcal{G}$  is convex and that  $\mathcal{G}$  has nonempty interior  $\mathcal{G}^\circ$ .*

*Let  $\mathcal{H}'$  be a set of policies which is given by the elements of  $\mathcal{H}$  which satisfy a linear budget*

constraint,  $\mathcal{H}' = \{h \in \mathcal{H} : \langle h, c \rangle = C\}$ .

Then the following two statements are equivalent:

- There are no  $h^a, h^b \in \mathcal{H}'$  such that  $h^a \succeq^{\mathcal{G}} h^b$ .
- $\lambda c$  is an element of  $\mathcal{G}^o$  for some  $\lambda \in \mathbb{R}$ .

The set of policy differences  $h^{ab}$  for pairs of policies satisfying the linear budget constraint of lemma 5 is given by the hyperplane  $c^\perp$ . The set of policy differences, such that the corresponding policies can be ordered, is given by the dual cone of  $\mathcal{G}$ , and its negative. Lemma 5 reflects the fact that the intersection of the hyperplane  $c^\perp$  with the dual cone of  $\mathcal{G}$  is equal to  $c^\perp$  if and only if  $\lambda c \in \mathcal{G}$  for some  $\lambda$ .

An interesting example where such a linear budget constraint arises is in the context of reallocations of individuals across groups, which might affect outcome distributions in the presence of social externalities, see Graham et al. (2008). In such a context, the feasible reallocations have to leave the population distribution of individual characteristics constant, see the discussion in section 5.1.

### 3 Nonlinear objective functions and local policy changes

Throughout section 2, we have assumed a linear objective function of the form  $SWF = E[Y]$ . This section generalizes the previous results to nonlinear objective functions of the form

$$SWF = \phi(F), \tag{39}$$

where  $F$  is the unconditional cumulative distribution function (CDF) of  $Y$ ,  $F(y) = P(Y \leq y)$ .

We will consider policies  $h$  which are close to some status quo policy  $h^0$ , and which result in outcome distributions  $F$  close to the status quo distribution  $F^0$ . This allows us to approximate, for the purpose of policy rankings,  $\phi(F)$  by  $\phi(F^0) + \partial\phi/\partial F \cdot (F - F^0)$ .<sup>6</sup> The continuous linear functional  $\partial\phi/\partial F$  is a functional derivative (or ‘‘Fréchet derivative’’) of  $\phi$ , which is presumed to exist, i.e.,

$$\lim_{F \rightarrow F^0} \frac{\|(\phi(F) - \phi(F^0)) - \partial\phi/\partial F \cdot (F - F^0)\|}{\|F - F^0\|} = 0,$$

where the norm  $\|F\|$  is defined as  $\|F\| = \sqrt{\int (dF/dF^0)^2 dF^0}$ . For a discussion of functional derivatives in statistics see van der Vaart (2000), chapter 20.

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<sup>6</sup>This idea is inspired by the approach of Firpo et al. (2009), applied in a different context. One way to interpret Firpo et al. (2009) is that they study inference on the effect of changing treatment assignment policies on functionals of the unconditional outcome distribution. They do this under an implicit assumption of conditional independence, which allows to point identify conditional average treatment effects.

Consider a family of treatment assignment policies indexed by  $\theta \in \mathbb{R}$ ,  $h(X, \theta) = P(D = 1|X, \theta)$ , where  $h(X, 0)$  equals the status quo policy  $h^0$ , and  $h$  is differentiable in  $\theta$ . It follows from the definition of differentiability that, for small enough  $\theta > 0$ , the family  $h^a$  is preferred to the family  $h^b$ , i.e.,  $SWF^{ab} := SWF^a - SWF^b > 0$ , if  $SWF_\theta^{ab} > 0$ . This derivative can be decomposed as

$$SWF_\theta^{ab} = \frac{\partial \phi}{\partial F} \cdot \frac{\partial F}{\partial h} \cdot h_\theta^{ab} > 0, \quad (40)$$

where

$$h_\theta^{ab}(X) = h_\theta^a(X) - h_\theta^b(X).$$

Note that the right hand side of equation (40) involves functional derivatives, of the scalar  $\phi$  with respect to the CDF  $F$ , and of  $F$  with respect to the treatment assignment policy function  $h$ . Both these derivatives are evaluated at the status quo distribution of outcomes  $F^0$  and the status quo treatment assignment policy  $h^0$ . If we assume that the linear functionals  $\partial\phi/\partial F$ ,  $\partial F(y)/\partial h$ , and  $\partial\phi/\partial h$  are  $L^2$ -continuous on their respective domains, then the Riesz representation theorem asserts the existence of a dual representation of these derivatives, i.e., the existence of functions  $IF$ ,  $g^y$ , and  $g^\phi$  such that

$$\begin{aligned} \phi_\theta &= \frac{\partial \phi}{\partial F} \cdot F_\theta = \int IF(y) dF_\theta(y) \\ F_\theta(y) &= \frac{\partial F}{\partial h} \cdot h_\theta = \langle h_\theta, g^y \rangle \\ \phi_\theta &= \frac{\partial \phi}{\partial h} \cdot h_\theta = \langle h_\theta, g^\phi \rangle. \end{aligned} \quad (41)$$

The function  $IF$  is the influence function of the parameter  $\phi$ , which plays a central role in the literature on semiparametric efficiency bounds, see for instance Tsiatis (2006). Note that  $\partial\phi/\partial F$  is bounded on the set of distribution functions if  $IF$  is bounded.

We already know the function  $g^y$  from section 2, it is given by the conditional average treatment effect on the conditional CDF  $F(Y|X)$ , evaluated at  $y$ :

$$\begin{aligned} g^y(X) &:= E[\mathbf{1}(Y^1 \leq y) - \mathbf{1}(Y^0 \leq y)|X] \\ &= F^{Y^1|X}(y|X) - F^{Y^0|X}(y|X). \end{aligned} \quad (42)$$

The arguments of section 2.1 imply that, for this definition of  $g^y$ , we indeed have  $\partial F/\partial h \cdot h_\theta = \langle h_\theta, g^y \rangle$ . Recall that  $\phi_\theta = \frac{\partial \phi}{\partial F} \cdot F_\theta = \frac{\partial \phi}{\partial F} \cdot \frac{\partial F}{\partial h} \cdot h_\theta$ . Using the dual representations of these derivatives, we get

$$\langle h_\theta, g^\phi \rangle = \int IF(y) dF_\theta(y) = \int IF(y) d\langle h_\theta, g^y \rangle = \left\langle h_\theta, \int IF(y) dg^y \right\rangle, \quad (43)$$

where the last equality follows from Fubini's theorem, allowing a change of the order of

integration with respect to  $Y$  and to  $X$ . Since equation (43) holds for all  $h_\theta$ , we get that

$$g^\phi = \int IF(y) dg^y.$$

A few **examples** might help to clarify ideas:

(i) If  $\phi = E[Y] = \int y dF(y)$ , we get

$$SWF_\theta^{ab} = \frac{\partial \phi}{\partial F} \cdot F_\theta^{ab} = \int y dF_\theta^{ab}(y) = \int y d\langle h_\theta^{ab}, g^y \rangle = \left\langle h_\theta^{ab}, \int y dg^y \right\rangle.$$

Thus in this case,  $g^\phi$  is given by the integral  $\int y dg^y$ , which equals  $g$  as it was defined in section 2.<sup>7</sup>

(ii) Take  $\phi = Var(Y) = E[Y^2] - E[Y]^2$ . Then

$$SWF_\theta^{ab} = \left\langle h_\theta^{ab}, \int y^2 dg^y \right\rangle - 2E[Y] \cdot \left\langle h_\theta^{ab}, \int y dg^y \right\rangle = \left\langle h_\theta^{ab}, \int (y^2 - 2E[Y] \cdot y) dg^y \right\rangle,$$

i.e.  $g^\phi = \int (y^2 - 2E[Y] \cdot y) dg^y$ . Note that, in contrast to the first example, the directional derivative of the variance does depend on the status quo distribution via  $E[Y]$ .

(iii) The  $\tau$ th quantile of  $Y$  is given by  $\phi = \inf\{y : F(y) \geq \tau\}$ . By the implicit function theorem, if  $Y$  is continuously distributed so that  $F$  is differentiable,  $\partial\phi/\partial\theta = -F_\theta(\phi)/F_y(\phi)$ , where  $F_y$  is the density of  $y$ . Thus

$$SWF_\theta^{ab} = -\frac{\langle h_\theta^{ab}, g^y \rangle}{F_y(y)} = \left\langle h_\theta^{ab}, \frac{-g^y}{F_y(y)} \right\rangle,$$

evaluated at  $y = \phi$ , so that  $g^\phi = \frac{-g^y}{F_y(y)}$ .

Consider now the preference relation

$$h^a \succ^{g^\phi} h^b :\Leftrightarrow \langle h^{ab}, g^\phi \rangle > 0,$$

(recall that  $SWF^{ab} > 0$ , for small  $\theta > 0$ , iff  $SWF_\theta^{ab} = \langle h_\theta^{ab}, g^\phi \rangle > 0$ ), and the corresponding identified preference relation  $\succ^{\mathcal{G}^\phi}$ , where  $\mathcal{G}^\phi$  is the identified set for  $g^\phi$ . We will in the rest of this section characterize  $\mathcal{G}^\phi$ . To do so, we need to start by deriving the identified set for  $g^y$ , jointly across  $y$ . This identified set then maps into the identified set for  $g^\phi$ .

Consider, first, instrumental variables, as discussed in appendix A.1, where we assume that a conditionally exogenous instrument  $Z$  is available, i.e.,  $Z \perp (Y^0, Y^1)|X$ . In this setup

<sup>7</sup>Note that, in order to keep notation simple, we are not normalizing influence functions to have mean zero in the examples. This is justified by the fact that  $dg^y$  and  $dF_\theta$  integrate to zero.

we can write

$$\begin{aligned}
g^y &= E[D|Z = z^1, X] \cdot P(Y \leq y|D = 1, Z = z^1, X) \\
&+ E[1 - D|Z = z^1, X] \cdot P(Y^1 \leq y|D = 0, Z = z^1, X) \\
&- E[1 - D|Z = z^0, X] \cdot P(Y \leq y|D = 0, Z = z^0, X) \\
&- E[D|Z = z^0, X] \cdot P(Y^0 \leq y|D = 1, Z = z^0, X),
\end{aligned} \tag{44}$$

This decomposition holds for any choice of  $z^0, z^1$ , in particular for  $z^1 = \operatorname{argmax}_z E[D|Z = z, X]$  and  $z^0 = \operatorname{argmin}_z E[D|Z = z, X]$ . All magnitudes in this expression are pinned down by the observable data distribution, except for the counterfactual distributions  $P(Y^1 \leq y|D = 0, Z = z^1, X)$  and  $P(Y^0 \leq y|D = 1, Z = z^0, X)$ , which are left unrestricted by the data. Denoting

$$\begin{aligned}
g^{mm,y} &= E[D|Z = z^1, X] \cdot P(Y \leq y|D = 1, Z = z^1, X) \\
&- E[1 - D|Z = z^0, X] \cdot P(Y \leq y|D = 0, Z = z^0, X),
\end{aligned}$$

and defining the counterfactual CDFs

$$\begin{aligned}
F^1 &= P(Y^1 \leq y|D = 0, Z = z^1, X) \\
F^0 &= P(Y^0 \leq y|D = 1, Z = z^0, X),
\end{aligned}$$

we can rewrite equation (44) as

$$\begin{aligned}
g^y &= g^{mm,y} + E[1 - D|Z = z^1, X] \cdot F^1 \\
&- E[D|Z = z^0, X] \cdot F^0.
\end{aligned}$$

In the IV setup discussed in appendix A.1, no restrictions on the counterfactual outcome distributions across  $X$  or across treatments are imposed. The counterfactual distributions are restricted, however, to have their support in  $[0, 1]$ .

Consider now the panel data setup as discussed in appendix A.2. In analogy to the IV case, the identified set for  $g^y$  is the set of functions which can be written as

$$\begin{aligned}
g^y &= g^{mm,y} + E[1 - M_1|X] \cdot F^1 \\
&- E[1 - M_0|X] \cdot F^0,
\end{aligned}$$

where now

$$g^{mm,y} = E[M_1|X] \cdot P(Y \leq y|M_1 = 1, X) \\ - E[M_0|X] \cdot P(Y \leq y|M_0 = 1, X),$$

and

$$F^1 = P(Y^1 \leq y|M_1 = 0, X) \\ F^0 = P(Y^0 \leq y|M_0 = 0, X).$$

We get the following lemma.

**Lemma 6 (Bounds on local policy effects)** *Define*

$$B^{1,\phi} = \sup_F \frac{\partial \phi}{\partial F} \cdot F \\ B^{0,\phi} = \inf_F \frac{\partial \phi}{\partial F} \cdot F, \quad (45)$$

where supremum and infimum are taken over the admissible cumulative distribution functions for counterfactual outcomes.

Then, in the case of instrumental variables as discussed in this section, we get a rectangular identified set for  $g^\phi$  which is bounded by

$$\bar{g}^\phi(X) = \frac{\partial \phi}{\partial F} \cdot g^{mm,y}(X) + E[1 - D|Z = z^1, X] \cdot B^{1,\phi} \\ - E[D|Z = z^0, X] \cdot B^{0,\phi} \\ \underline{g}^\phi(X) = \frac{\partial \phi}{\partial F} \cdot g^{mm,y}(X) + E[1 - D|Z = z^1, X] \cdot B^{0,\phi} \\ - E[D|Z = z^0, X] \cdot B^{1,\phi},$$

where  $z^1 = \operatorname{argmax}_z E[D|Z = z, X]$  and  $z^0 = \operatorname{argmin}_z E[D|Z = z, X]$ .

In the case of the panel data setup as discussed in this section, we get a rectangular identified set for  $g^\phi$  which is bounded by

$$\bar{g}^\phi(X) = \frac{\partial \phi}{\partial F} \cdot g^{mm,y}(X) + E[1 - M_1|X] \cdot B^{1,\phi} \\ - E[1 - M_0|X] \cdot B^{0,\phi} \\ \underline{g}^\phi(X) = \frac{\partial \phi}{\partial F} \cdot g^{mm,y}(X) + E[1 - M_1|X] \cdot B^{0,\phi} \\ - E[1 - M_0|X] \cdot B^{1,\phi}.$$

Note that the bounds in equation (45) need not be finite. There are however functionals  $\phi$  with bounded derivatives  $\partial\phi/\partial F$  on the set of admissible cumulative distribution functions, or equivalently, with bounded influence function. In fact, those are exactly the parameters  $\phi$  considered robust in the literature on robust statistics, see for instance Huber (1996). We get

**Corollary 1 (Identification and robust statistics)** *Under the assumptions of of lemma 6, the following holds:*

*The observable data distribution implies finite bounds on  $g^\phi$ , if and only if  $\partial\phi/\partial F$  is bounded on the admissible support for the counterfactual outcome distributions.*

*The observable data distribution therefore implies a non-trivial identified preference ordering  $\succ^{\mathcal{G}^\phi}$  only if  $\partial\phi/\partial F$  is bounded.*

*Furthermore, the observable data distribution implies a non-trivial identified preference ordering based on  $SWF = \phi$  (rather than its linear approximation) for all neighborhoods of  $h^0$  under the same conditions.*

Let us again consider the three **examples** of functionals  $\phi$ .

(i) If  $\phi = E[Y]$ , we have  $\partial\phi/\partial F \cdot \tilde{F} = \int y d\tilde{F}(y)$ . This is unbounded for general outcome distributions. If  $Y$  is known to have its support in  $[0, 1]$ ,  $\partial\phi/\partial F$  is bounded by  $[0, 1]$  as well. The extrema  $B^{1,\phi}$  and  $B^{0,\phi}$  are achieved by concentrating the support of the counterfactual data distribution at 0 and 1, respectively.

(ii) If  $\phi = Var(Y) = E[Y^2] - E[Y]^2$ , we have  $\partial\phi/\partial F \cdot \tilde{F} = \int (y^2 - 2E[Y] \cdot y) d\tilde{F}$ . As in the first example, this is unbounded for general outcome distributions. If  $Y$  is known to have its support in  $[0, 1]$ ,  $\partial\phi/\partial F$  is bounded by  $B^{1,\phi} = \max\{1 - 2E[Y], 0\}$  and  $B^{0,\phi} = -E[Y]^2$ . To see this note that the integrand  $(y^2 - 2E[Y] \cdot y)$  is convex. The supremum is thus achieved by concentrating the support of  $\tilde{F}$  at 0 or 1, the infimum is achieved by concentrating the support at  $y = E[Y]$ .

(iii) If  $\phi = \inf\{y : F(y) \geq \tau\}$  and  $F$  is differentiable, we have  $\partial\phi/\partial F \cdot \tilde{F} = -\tilde{F}(\phi)/F_y(\phi)$ , and thus  $B^{1,\phi} = 0$ ,  $B^{0,\phi} = -1/F_y(\phi)$ . Quantiles thus have bounded influence functions even in the absence of a-priori restrictions on the counterfactual outcome distribution; they are “robust”. As a consequence, nonparametric instrumental variables or panel data setups give bounded confidence sets for the effect of policy changes on unconditional quantiles of the outcome distribution even for unbounded outcomes.

## 4 Optimal policies

Section 2 studied the problem of deriving a social preference relation  $\succeq^{\mathcal{G}}$  on the set of feasible policies  $\mathcal{H}'$  from an identified set  $\mathcal{G}$  for conditional average treatment effects  $g$ . This section studies optimal policies. These depend on the objective function (which was discussed in section 3, and is again left implicit here), as well as on the set of feasible policies. Denote

the optimal policy  $h$  among the feasible policies  $\mathcal{H}'$  given the conditional average treatment effect  $g$  by

$$h^*(g) := \operatorname{argmax}_{h \in \mathcal{H}'} \langle h, g \rangle, \quad (46)$$

where we assume throughout that  $h^*$  is well defined. Note that in general  $h^*$  is a correspondence. Note also the similarity of this problem to the classic profit maximization problem, where  $g$  corresponds to prices and  $\mathcal{H}'$  to the production possibility set. The identified set for the optimal policy  $h^*$  is given by

$$\mathcal{H}^*(\mathcal{G}) = \bigcup_{g \in \mathcal{G}} \operatorname{argmax}_{h \in \mathcal{H}'} \langle h, g \rangle = \bigcup_{g \in \mathcal{G}} h^*(g). \quad (47)$$

We can also define a value function giving the maximum welfare attainable,

$$V(g) = \max_{h \in \mathcal{H}'} \langle h, g \rangle = \langle h^*(g), g \rangle. \quad (48)$$

Note that we have normalized welfare under the policy  $h = 0$  to  $V(0) = 0$ .  $V(g)$  is a convex function of  $g$ , and homogeneous of degree one. Furthermore, if  $h^*(g)$  is a singleton and  $X$  has finite support, then  $\partial V / \partial g = h^*(g)$ ; see for instance Mas-Colell et al. (1995), chapter 5.

If  $\mathcal{H}'$  is unconstrained, an optimal treatment assignment rule is given by<sup>8</sup>

$$h^*(g) = \mathbf{1}(g > 0). \quad (49)$$

Let

$$j(y) = y \cdot \mathbf{1}(y > 0). \quad (50)$$

The value function for unconstrained  $\mathcal{H}'$  can be written as

$$V(g) = \langle h^*(g), g \rangle = \langle \mathbf{1}(g > 0), g \rangle = E_X[j(g(X))]. \quad (51)$$

If  $\mathcal{H}'$  corresponds to the set of policies in  $\mathcal{H}$  satisfying a linear budget constraint of the form  $\langle h, c \rangle = C$ , the Kuhn-Tucker conditions for optimality of  $h^*$  imply

$$h^*(g) = \mathbf{1}(g - \lambda c > 0) + \kappa \mathbf{1}(g = \lambda c), \quad (52)$$

where  $\kappa \in [0, 1]$ . Note that, if  $c > 0$ , we can rewrite this as

$$h^*(g) = \mathbf{1}(g/c > \lambda) + \kappa \mathbf{1}(g/c = \lambda).$$

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<sup>8</sup>In a slight abuse of notation we use  $h^*$  to denote both the set of maximizers of *SWF*, and an element thereof.

The optimal policy assigns treatment 1 for all  $X$ , such that the “benefit to cost ratio”  $g/c$  exceeds a threshold  $\lambda$ . The threshold  $\lambda$ , and  $\kappa$ , are determined by the constraint  $\langle h, c \rangle = C$ .

Suppose we chose  $h$  believing that conditional average treatment effects are given by  $g'$ , whereas they are actually given by  $g$ . The welfare we lose relative to the optimum attainable is given by the regret function

$$R(g, g') = \langle h^*(g) - h^*(g'), g \rangle = V(g) - \langle h^*(g'), g \rangle. \quad (53)$$

Subsection 4.1 characterizes maximum regret, given an identified set  $\mathcal{G}$  for  $g$ . This provides a natural decision-theoretic measure for the quality of identification approaches; Starting from a given identification approach, it provides an upper bound on what we would be willing to pay in order to learn the true  $g$ . This bound is independent of the decision rule used to choose a policy from  $\mathcal{H}^*(\mathcal{G})$ .

Subsection 4.2 characterizes expected regret under a Bayesian prior, where policy is chosen to maximize expected welfare.

#### 4.1 Maximum regret

This subsection characterizes and bounds maximum regret,

$$MR(\mathcal{G}) := \max_{g, g' \in \mathcal{G}} R(g, g') = \max_{g \in \mathcal{G}} \left[ V(g) - \min_{h \in \mathcal{H}^*(\mathcal{G})} \langle h, g \rangle \right]. \quad (54)$$

If  $MR(\mathcal{G}) < \epsilon$ , then the maximal regret from choosing any policy which is optimal for some  $g'$  in the identified set, relative to the optimal policy  $h^*(g)$ , is less than or equal  $\epsilon$ . Put differently, this is a condition guaranteeing that our “willingness to pay” for gaining knowledge of  $g$ , rather than just the identified set  $\mathcal{G}$ , is no more than  $\epsilon$ . This bound is independent of the decision rule applied to choose among elements of  $\mathcal{H}^*(\mathcal{G})$ . Note however that there might be decision rules which do not choose an element of  $\mathcal{H}^*(\mathcal{G})$ ; for such a rule the maximum regret bound need not apply. The bound does apply to Bayesian rules, maximin, and minimax regret.

**Lemma 7 (Bounds on regret, unconstrained policy sets)** *Suppose that the identified set  $\mathcal{G}$  is rectangular, with upper and lower bounds  $\bar{g}$  and  $\underline{g}$ , and width  $w = \bar{g} - \underline{g}$ , and suppose the set of feasible policies  $\mathcal{H}'$  is unrestricted. Then*

$$MR(\mathcal{G}) = E \left[ \mathbf{1}(\bar{g} > 0 > \underline{g}) \cdot \max(\bar{g}, -\underline{g}) \right] \quad (55)$$

and in particular

$$MR(\mathcal{G}) \leq E[w]. \quad (56)$$

Furthermore  $MR(\mathcal{G}) > E[w] - \delta$  for all  $\delta > 0$  and some rectangular  $\mathcal{G}$  of width  $w$ .

Equation (56) states the intuitive result that our maximal willingness to pay in order to gain knowledge of  $g$  is no larger than the size of the identified set,  $E[w]$ . Recall that for instrumental variables,  $w$  is given by  $w^{IV}(X) = 1 - E[D|X, Z = z^1] + E[D|X, Z = z^0]$ , measuring the strength of the instrument  $Z$ . Lemma 7 implies a proportional willingness to pay for increasing the strength of instruments.

**Lemma 8 (Bounds on regret, linearly constrained policy sets)** *Suppose that the identified set  $\mathcal{G}$  is rectangular, with upper and lower bounds  $\bar{g}$  and  $\underline{g}$ , and width  $w = \bar{g} - \underline{g}$ , and suppose the set of feasible policies  $\mathcal{H}'$  is subject to a constraint of the form  $\langle c, h \rangle = C$ . Then*

$$MR(\mathcal{G}) \leq E \left[ w \cdot \mathbf{1} \left( \frac{w}{c} \geq \omega \right) \right], \quad (57)$$

where  $\omega$  is such that

$$E \left[ c \cdot \mathbf{1} \left( \frac{w}{c} \geq \omega \right) \right] = 2C.$$

Furthermore  $MR > E \left[ w \cdot \mathbf{1} \left( \frac{w}{c} \geq \omega \right) \right] - \delta$  for all  $\delta > 0$  and some rectangular  $\mathcal{G}$  of width  $w$ . In the case  $c \equiv 1$  (constant cost of treatment), condition (57) specializes to

$$MR(\mathcal{G}) \leq E[w \cdot \mathbf{1}(w \geq \omega)], \quad (58)$$

where  $\omega = Q^{1-2C}(w)$  is the  $1 - 2C$ th quantile of  $w$ . In particular, the bound on maximum regret  $MR(\mathcal{G})$  is increasing in  $C$ , and is equal to the bound of the unconstrained case for  $C \geq E[c]/2$ .

Comparing lemma 8 to lemma 7 shows that more restricted policy sets lead to lower maximum regret. Increasing the share of units which can be treated increases maximum regret, however only to the point where half the units can be treated.

It is tempting to think about maximum regret by considering the “distance function”

$$MR(g, g') = \max(R(g, g'), R(g', g)),$$

which gives  $MR(\mathcal{G}) = \max_{g, g' \in \mathcal{G}} MR(g, g')$ , i.e., maximum regret is the diameter of the set  $\mathcal{G}$  with respect to  $MR$ . However, while this function is symmetric by construction, it is not positive definite; consider  $g' = 2g$ , then  $R(g, g') = 0$ .  $MR$  does not satisfy the triangle inequality either; consider unconstrained  $\mathcal{H}'$  and constant  $g^1, g^2, g^3$  such that  $-g^2 < g^1 < 0 < g^2 < g^3$ . Then  $MR(g^1, g^2) + MR(g^2, g^3) = g^2 + 0 > 0 = MR(g^2, g^3)$ .

A particular principle to choose among policies in the presence of ambiguity, advocated among others by Savage (1951) and Manski (2011), is given by “minimax regret”. In the

present setup,

$$h^{mmr} = \operatorname{argmin}_{h \in \mathcal{H}'} \max_{g \in \mathcal{G}} (V(g) - \langle h, g \rangle). \quad (59)$$

The following lemma adapts equation (14) from Manski (2011) to the present context.

**Lemma 9 (Minimax regret)** *Suppose that the identified set  $\mathcal{G}$  is rectangular, with upper and lower bounds  $\bar{g}$  and  $\underline{g}$ , and width  $w = \bar{g} - \underline{g}$ , and suppose the set of feasible policies  $\mathcal{H}'$  is unrestricted. Then*

$$h^{mmr} = \mathbf{1}(\bar{g} > 0 > \underline{g}) \cdot \frac{\bar{g}}{w} + \mathbf{1}(\underline{g} > 0) \quad (60)$$

and

$$\max_{g \in \mathcal{G}} (V(g) - \langle h^{mmr}, g \rangle) = E \left[ \mathbf{1}(\bar{g} > 0 > \underline{g}) \cdot \frac{-\bar{g} \cdot \underline{g}}{w} \right] \quad (61)$$

$$\leq E[w]/4. \quad (62)$$

The bound (62) is attained for  $\underline{g} = -w/2$ ,  $\bar{g} = w/2$ .

Note that the bound on minimax regret derived in lemma 9,  $E[w]/4$ , is proportional to the bound on maximum regret in lemma 7,  $E[w]$ .

## 4.2 Bayesian expected regret

This section studies Bayesian optimal decision rules and associated expected regret. The approach taken here is ‘‘Bayesian in the limit’’, i.e., posteriors are conditional on knowledge of the observable data distribution  $\mathcal{P}$ , rather than conditional on just a sample from that distribution. In particular, the posterior distribution for  $g$  has its support on the identified set  $\mathcal{G}$ . The optimal policy conditional on  $\mathcal{P}$  is given by

$$h^B(\mathcal{P}) = \operatorname{argmax}_{h \in \mathcal{H}'} E[\langle h, g \rangle | \mathcal{P}] = \operatorname{argmax}_{h \in \mathcal{H}'} \langle h, E[g | \mathcal{P}] \rangle,$$

where the expectations are taken over the posterior distribution of  $g$  given  $\mathcal{P}$ . Let  $\hat{g} := E[g | \mathcal{P}]$  be the posterior expectation of  $g$ . We get

$$h^B = h^*(\hat{g}). \quad (63)$$

The corresponding (posterior) expected welfare is given by

$$V^B(\mathcal{P}) = \max_{h \in \mathcal{H}'} E[\langle h, g \rangle | \mathcal{P}] = \langle h^*(\hat{g}), \hat{g} \rangle = V(\hat{g}). \quad (64)$$

Note that equations (63) and (64) hold because of the linearity of our objective function in  $g$ . The posterior expected regret, conditional on  $\mathcal{P}$ , of a Bayesian policy maker is thus given by

$$\begin{aligned}
ER(\mathcal{P}) &:= E[R(g, \hat{g}) | \mathcal{P}] \\
&= E[V(g) | \mathcal{P}] - V^B(\mathcal{P}) \\
&= E[V(g) | \mathcal{P}] - V(E[g | \mathcal{P}]) \\
&= E[\langle h^*(g) - h^*(\hat{g}), g \rangle | \mathcal{P}].
\end{aligned} \tag{65}$$

Ex ante expected regret, if decisions are based on  $\mathcal{P}$ , equals

$$ER^{\mathcal{P}} = E[ER(\mathcal{P})] = E[V(g) - V(\hat{g})] \tag{66}$$

The following lemma states the unsurprising fact that ex-ante expected welfare is increasing in the amount of information revealed by the observed data distribution. This follows immediately from equation (65), once we recall that  $V$  is convex, and apply Jensen's inequality.

**Lemma 10 (More information is better, Bayesian version)** *Suppose the distribution  $\mathcal{P}$  can be written as a function of the distribution  $\mathcal{O}$ . Then  $ER^{\mathcal{P}} \geq ER^{\mathcal{O}}$ .*

Recall from equation (51) that, in the case of unrestricted policy sets  $\mathcal{H}'$ , the value function is given by  $V(g) = E_X[j(g(X))]$ , where  $j(y) = y \cdot \mathbf{1}(y > 0)$ . By equation (65), then,

$$\begin{aligned}
ER(\mathcal{P}) &= E_g[V(g) | \mathcal{P}] - V(\hat{g}) \\
&= E_g[E_X[j(g(X)) | \mathcal{P}] - E_X[j(\hat{g}(X))]] \\
&= E_g[E_X[j(g(X)) - j(\hat{g}(X)) | \mathcal{P}]] \\
&= E_g[E_X[(\mathbf{1}(g(X) > 0) - \mathbf{1}(\hat{g}(X) > 0)) \cdot g(X) | \mathcal{P}]] \\
&= E_g[E_X[j(-g(X) \cdot \text{sign}(\hat{g}(X))) | \mathcal{P}]].
\end{aligned} \tag{67}$$

With a trapezoidal posterior, derived from uniform priors in lemma 12 in the appendix, we get the following result on expected regret. In this result,  $\gamma$  is a function of  $X$  ranging from 0 for a uniform posterior (given  $X$ ) to 1 for a symmetric triangular posterior. Given uniform priors, the parameter  $\gamma$  reflects the (a)symmetry of uncertainty about counterfactual outcomes. If the mean of one of the potential outcome distributions is point identified, then  $\gamma$  equals 0; if the identified sets for the counterfactual means for either treatment are of equal size, then  $\gamma$  equals 1/2.

**Lemma 11** *Suppose the posterior of  $g$  is given by a symmetric trapezoidal distribution on a rectangular  $\mathcal{G}$ , as in lemma 12, and that the set of feasible policies  $\mathcal{H}'$  is unconstrained.*

Then posterior expected regret is given by

$$ER(\mathcal{P}) = E_X \left[ \frac{1}{6w^2\gamma(1-\gamma)} \left( j(w/2 - |\hat{g}|)^3 - j((1/2 - \gamma)w - |\hat{g}|)^3 \right) \right].$$

## 5 Applications

### 5.1 The effect of segregation

Graham et al. (2008) discuss identification and inference on the effect of policies which reallocate individuals across groups. Such reallocations do have an effect if there are social spillovers. Identification in Graham et al. (2008) is based on a “double randomization” assumption, which allows them to focus on the relationship between feasible policies and parameters of interest. This subsection reformulates the problem in terms of the setup we have used so far, and discusses identification of the effect of marginal reallocative policies in the absence of such double randomization.

Suppose we are interested in the impact of gender segregation or integration, across school classes, on average test scores. Assume for simplicity that classes are of constant size  $n$ , and that a student’s test score is a function of the number of girls in a class,  $n_g$ , as well as other factors. An educational policymaker might be interested in the effect on average test scores of reallocating students across classes, based on gender. A marginal such reallocation might first remove one student from each class, leaving  $n - 1$  students and  $n'_g$  girls. It might then reassign students to classes based on their gender and based on the number  $n'_g$  of girls already in a class. The share of classes to which an additional girl is assigned has to equal the share of girls among the students to be assigned,  $s_g$ .

To fit this into the framework of the present paper, denote average test score in a class by  $Y$ ; the share of girls in a class before assignment of the additional student by  $X = n'_g$ ; take treatment  $D$  to equal 1 if  $n_g = n'_g + 1$ , and  $D = 0$  if  $n_g = n'_g$ ; and consider policies subject to the budget constraint  $E[h(X)] = E[s_g]$ .

Suppose we have panel data on class composition and average (normalized) test scores. We might be willing to make the marginal stationarity assumption discussed in appendix A.2, which asserts in this context that changes of composition over time, for a given class, are independent of changes of other determinants of test scores. Under this assumption, we can partially identify the effect of composition on scores.

## 6 Conclusion

The goal of this paper is to explore the frontier in the trade-off between the twin goals of recognition of the limits of our knowledge on the one hand, and the necessity to give informed

policy recommendations on the other hand. Philosophically speaking, the goal is to reconcile a position of Scepticism, which refrains from making strong a priori assumptions, and of Pragmatism, which asserts that relevant empirical parameters are those having implications for policy choice. Two sets of questions were discussed in particular: Under what conditions and to what extent is the welfare ranking of policies identified? And what would be the welfare gain from increasing our knowledge of underlying structural parameters? The answer to these questions depends on the interaction of the identified set, the feasible policy set, and the objective function. This paper discusses these questions in the context of policies allocating a binary treatment based on observable covariates, under partial identification of conditional average treatment effects and with possibly restricted sets of feasible policies.

## Appendix A: Nonparametric partial identification of conditional average treatment effects

In this section, we will review some identification results from the literature and relate them to our general framework. These results can be understood to combine “reduced form” estimates of causal effects<sup>9</sup> with a priori bounds on unobserved potential outcomes<sup>10</sup>, in order to obtain bounds on average treatment effects. The latter are invariant to identification approaches and policies, hence “structural”.

### A.1 Instrumental variables

Recall the general setup of section 1.1,  $Y = f(X, D, \epsilon)$ ,  $D \in \{0, 1\}$  and  $Y \in [0, 1]$ , where we denote the potential outcomes by  $Y^d = f(X, d, \epsilon)$  for  $d = 0, 1$ . The nonparametric instrumental variables setup, as discussed for instance in Manski (2003) chapter 2, is based on the additional assumption that a conditionally exogenous instrument  $Z$  is available:

$$Z \perp \epsilon | X. \tag{68}$$

Let

$$\begin{aligned} \overline{Y^1}(X, Z) &= E[YD + (1 - D)|X, Z] \\ \underline{Y^1}(X, Z) &= E[YD|X, Z] \\ \overline{Y^0}(X, Z) &= E[D + Y(1 - D)|X, Z] \\ \underline{Y^0}(X, Z) &= E[Y(1 - D)|X, Z]. \end{aligned} \tag{69}$$

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<sup>9</sup>i.e., causal effects for subpopulations, or “local average treatment effects”, see Imbens and Angrist (1994).  
<sup>10</sup>for the “non-compliers” in the case of IV, for the “stayers” in the case of panel data

These are bounds on the conditional average structural functions  $E[Y^d|X, Z]$ . They hold because of the bounds on the support of  $Y$ . Next, define

$$\begin{aligned}\overline{Y^1}(X) &= \min_z \overline{Y^1}(X, z) \\ \underline{Y^1}(X) &= \max_z \underline{Y^1}(X, z) \\ \overline{Y^0}(X) &= \min_z \overline{Y^0}(X, z) \\ \underline{Y^0}(X) &= \max_z \underline{Y^0}(X, z).\end{aligned}\tag{70}$$

These are tight bounds on the conditional average structural functions  $E[Y^d|X]$ . They hold because the exogeneity of  $Z$  implies  $E[Y^d|X, Z] = E[Y^d|X]$ . Denote the optimizers of these expressions by  $z^{1u}, z^{1l}, z^{0u}$  and  $z^{0l}$ , where we assume that these optimizers exist. This is guaranteed in particular if the support of  $Z$  is finite.

In general these  $z$  are functions of  $X$ . If, however,  $D$  is generated by a first stage relationship of the form

$$D = d(X, Z, \eta),\tag{71}$$

and

$$Z \perp (\epsilon, \eta)|X,$$

where  $d$  is monotonically increasing in  $Z$ , then the optimizers  $z^{1u}, z^{1l}, z^{0u}$  and  $z^{0l}$  are constant in  $X$  and  $z^{1u} = z^{1l}$  and  $z^{0u} = z^{0l}$ . To see this, consider  $z' > z''$ . Exogeneity and monotonicity imply

$$\begin{aligned}\overline{Y^1}(X, z') - \overline{Y^1}(X, z'') &= E[YD + (1 - D)|X, Z = z'] - E[YD + (1 - D)|X, Z = z''] \\ &= E[Y^1 d(X, z', \eta) + (1 - d(X, z', \eta))|X] \\ &\quad - E[Y^1 d(X, z'', \eta) + (1 - d(X, z'', \eta))|X] \\ &= E[(Y^1 - 1)(d(X, z', \eta) - d(X, z'', \eta))|X] \leq 0,\end{aligned}\tag{72}$$

and

$$\begin{aligned}\underline{Y^1}(X, z') - \underline{Y^1}(X, z'') &= E[YD|X, Z = z'] - E[YD|X, Z = z''] \\ &= E[Y^1(d(X, z', \eta) - d(X, z'', \eta))|X] > 0.\end{aligned}\tag{73}$$

Similar arguments hold for  $\overline{Y^0}(X, Z)$  and  $\underline{Y^0}(X, Z)$ . It follows that  $z^{1u} = z^{1l} = z^1 = \max z$  and  $z^{0u} = z^{0l} = z^0 = \min z$ .

The worst-case  $g$  is given by

$$g^{mm}(X) = \underline{Y}^1(X) - \underline{Y}^0(X) = E[YD|X, Z = z^1] - E[Y(1 - D)|X, Z = z^0]. \quad (74)$$

In order to apply lemma 3 in the present setup, we can rewrite

$$\begin{aligned} \bar{g}(X) &= \overline{Y}^1(X) - \underline{Y}^0(X) = g^{mm}(X) + (\overline{Y}^1(X) - \underline{Y}^1(X)) \\ &= g^{mm}(X) + E[1 - D|X, Z = z^1] \\ \underline{g}(X) &= \underline{Y}^1(X) - \overline{Y}^0(X) = g^{mm}(X) - (\overline{Y}^0(X) - \underline{Y}^0(X)) \\ &= g^{mm}(X) - E[D|X, Z = z^0]. \end{aligned} \quad (75)$$

This implies

$$\bar{g}(X) - \underline{g}(X) = 1 - E[D|X, Z = z^1] + E[D|X, Z = z^0]. \quad (76)$$

The width of the identified interval is given by the function

$$\begin{aligned} w^{IV}(X) &:= 1 - (\max_z E[D|X, Z = z] - \min_z E[D|X, Z = z]) \\ &= 1 - E[D|X, Z = z^1] + E[D|X, Z = z^0]. \end{aligned} \quad (77)$$

The function  $w^{IV}(X)$  is decreasing in the strength of the instrument given  $X$ , from 1 for irrelevant instruments to 0 for instruments perfectly predictive of  $D$  given  $X$ . We can now apply lemma 3 with  $w = w^{IV}$ .

If  $h^a, h^b$  are deterministic policies, we can rewrite

$$\langle |h^{ab}|, w^{IV} \rangle = \left( 1 - E[D|\mathcal{X}^{ab}, Z = z^1] + E[D|\mathcal{X}^{ab}, Z = z^0] \right) P(\mathcal{X}^{ab}), \quad (78)$$

which allows to rewrite the sufficient condition for identification of  $\text{sign}(SWF^{ab})$  given in lemma 3 as

$$E[D|\mathcal{X}^{ab}, Z = z^1] - E[D|\mathcal{X}^{ab}, Z = z^0] > 1 - |SWF^{ab}| / P(\mathcal{X}^{ab}). \quad (79)$$

## A.2 Panel data

Assume we observe panel data  $(X_i, D_{it}, Y_{it})$  for  $t = 1, \dots, T$ , where

$$Y_{it} = f(X_i, D_{it}, \epsilon_{it}), \quad (80)$$

and make the ‘‘marginal stationarity’’ assumption

$$\epsilon_{it}|X_i, D_i^T \sim \epsilon_{i1}|X_i, D_i^T \quad (81)$$

where  $D_i^T = (D_{i1}, \dots, D_{iT})$ . This is the main identifying assumption maintained in Graham and Powell (2010) and in the nonparametric part of Chernozhukov et al. (2010). Similar to before we use the potential outcome notation

$$Y_{it}^d = f(X_i, d, \epsilon_{it}).$$

In a slight abuse of notation, write  $d \in D^T$  iff there is some  $t \leq T$  such that  $D_t = d$ . We can define conditional average structural functions<sup>11</sup>

$$ASF^d(X) = E[Y^d|X] \tag{82}$$

and

$$ASF^d(X, D^T) = E[Y^d|X, D^T] = E[f(X_i, d, \epsilon_{it})|X, D^T]. \tag{83}$$

Note that these expressions are well defined by the marginal stationarity assumption. Furthermore,  $ASF^d$  is identified for  $d \in D^T$  by

$$ASF^d(X, D^T) = E[Y_{t_d}|X, D^T], \tag{84}$$

where  $t_d$  is such that  $D_{t_d} = d$ . W.l.o.g. choose  $t_d$  to be the smallest such  $t$ , and  $t_d = T + 1$  if there is no such  $t$ . Let  $M_d = \mathbf{1}(d \in D^T)$  be an indicator for  $d$  occurring for a given unit of observation. The data place no further restriction on  $ASF^d$ , giving the bounds  $ASF^d(X) \in [\underline{Y}^d(X), \overline{Y}^d(X)]$  where

$$\begin{aligned} \overline{Y}^d(X) &= E[Y_{t_d}M_d + (1 - M_d)|X] \\ \underline{Y}^d(X) &= E[Y_{t_d}M_d|X] \end{aligned} \tag{85}$$

The worst-case  $g$  is given by

$$g^{mm}(X) = \underline{Y}^1(X) - \underline{Y}^0(X) = E[Y_{t_1}M_1|X] - E[Y_{t_0}M_0|X]. \tag{86}$$

We can again use lemma 3 to provide sufficient conditions for condition (12) to hold. In particular

$$\begin{aligned} \bar{g}(X) &= \overline{Y}^1(X) - \underline{Y}^0(X) = g^{mm}(X) + (\overline{Y}^1(X) - \underline{Y}^1(X)) \\ &= g^{mm}(X) + E[(1 - M_1)|X] \\ \underline{g}(X) &= \underline{Y}^1(X) - \overline{Y}^0(X) = g^{mm}(X) - (\overline{Y}^0(X) - \underline{Y}^0(X)) \\ &= g^{mm}(X) - E[(1 - M_0)|X]. \end{aligned} \tag{87}$$

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<sup>11</sup>omitting the unit subscript  $i$  for the rest of this section

This implies that the width of the identified interval is given by the function

$$w^M(X) := \bar{g}(X) - \underline{g}(X) = E[(1 - M_1)|X] + E[(1 - M_0)|X]. \quad (88)$$

Note that  $M_0 + M_1 \geq 1$ , thus  $w^M \leq 1$ . The function  $w^M(X)$  is decreasing in the informativeness of the panel data, from 1 for completely uninformative data to 0 for data point identifying  $ASF(X)$ .

If  $h^a, h^b$  are deterministic policies, we can rewrite

$$\langle h^{ab}, w^M \rangle = E[(1 - M_1) + (1 - M_0)|\mathcal{X}^{ab}]P(\mathcal{X}^{ab}), \quad (89)$$

which allows to rewrite the sufficient condition for identification of  $\text{sign}(SWF^{ab})$  given in lemma 3 as

$$E[M_1 + M_0|\mathcal{X}^{ab}] > 2 - |SWF^{ab}|/P(\mathcal{X}^{ab}). \quad (90)$$

## Appendix B: Bayesian priors

This section proposes simple priors for the nonparametric instrumental variable and panel data setups discussed in this paper. Priors for nonparametric models are an active area of research, for an overview see Ghosh and Ramamoorthi (2003). Here, only simple uniform priors over the means of counterfactual distributions are considered. These priors allow to derive tractable expressions for optimal policies and expected welfare.

### B.1 Instrumental variables

Bayesian analysis of instrumental variables with binary treatment and instrument is discussed in Imbens and Rubin (1997). They are mainly interested in inference on the local average treatment effect (LATE), which is point identified. We are in contrast interested in identification of the average treatment effect (ATE). This section discusses the posterior distribution of the conditional average treatment effect  $g$ , conditional on the observable data distribution. Recall from appendix A.1 that, under the instrumental variables assumptions maintained in that section, we can write  $g$  as

$$g = g^{mm} + E[1 - D|X, Z = z^1] \cdot u^1 - E[D|X, Z = z^0] \cdot u^2, \quad (91)$$

where

$$\begin{aligned} u^1 &= E[Y^1|X, Z = z^1, D = 0] \\ u^2 &= E[Y^0|X, Z = z^0, D = 1] \end{aligned} \quad (92)$$

and

$$g^{mm} = E[YD|X, Z = z^1] - E[Y(1 - D)|X, Z = z^0]. \quad (93)$$

The observable data distribution pins down  $g^{mm}$ ,  $E[1 - D|X, Z = z^1]$ , and  $E[D|X, Z = z^0]$ . It is uninformative about  $u^1, u^2$ . The bounds reviewed in appendix A.1 arise if we set  $u^1, u^2$  equal to the bounds of the support of  $Y$ . By equation (91), the posterior distribution is thus a function of the posterior distribution of  $u^1$  and  $u^2$ .

Particular choices of priors then lead to tractable posteriors, as in the following lemma. This lemma assumes that our prior over  $u^1$  and  $u^2$  is uniform on their support, and that  $u^1(X), u^2(X)$  are independent of the observable data distribution as well as of  $u^1(X'), u^2(X')$  for  $X' \neq X$ .

**Lemma 12** *Suppose that  $X$  has finite support and that the prior satisfies*

$$P(u^1, u^2 | \mathcal{P}) = \prod_x \mathbf{1}(0 \leq u^1(x), u^2(x) \leq 1). \quad (94)$$

Let

$$\gamma = \frac{\min(E[D|X, Z = z^0], E[1 - D|X, Z = z^1])}{E[D|X, Z = z^0] + E[1 - D|X, Z = z^1]}.$$

Then the posterior of  $g(X)$  is independent across  $X$  and symmetric trapezoidal on  $[\underline{g}(X), \bar{g}(X)]$ , i.e., its density is equal to

$$p(g(X) = y | \mathcal{P}) = \frac{1}{w(1 - \gamma)} \cdot \begin{cases} \frac{y - \underline{g}(X)}{w\gamma} & \text{for } y \in \underline{g}(X) + w \cdot [0, \gamma] \\ 1 & \text{for } y \in \underline{g}(X) + w \cdot [\gamma, 1 - \gamma] \\ \frac{\bar{g}(X) - y}{w\gamma} & \text{for } y \in \underline{g}(X) + w \cdot [1 - \gamma, 1]. \end{cases}$$

In particular, the posterior expectation of  $g$  is given by

$$\hat{g} = \frac{1}{2}(\underline{g} + \bar{g}).$$

## B.2 Panel data

In section A.2 we showed that, under the panel model assumptions maintained in that section, we can write

$$g = g^{mm} + E[(1 - M_1)|X] \cdot u^1 - E[(1 - M_0)|X] \cdot u^2, \quad (95)$$

where

$$\begin{aligned} u^1 &= E[Y^1|X, M_1 = 0] \\ u^2 &= E[Y^0|X, M_0 = 0] \end{aligned} \tag{96}$$

and

$$g^{mm} = E[Y_{t_1} M_1|X] - E[Y_{t_0} M_0|X]. \tag{97}$$

The observable data distribution pins down  $g^{mm}$ ,  $E[1 - D|X, Z = z^1]$ , and  $E[Y^0|X, Z = z^0, D = 1]$ . It is uninformative about  $u^1, u^2$ . The bounds reviewed in section A.1 arise if we set  $u^1, u^2$  equal to the bounds of the support of  $Y$ . By equation (95), the posterior distribution is thus a function of the posterior distribution of  $u^1$  and  $u^2$ .

## Appendix C: Dependence of the policy ranking on distributional preferences

This paper has taken the social objective function  $SWF = E[Y]$  as given, where  $Y$  was some transformation of the observable data. Now we shall make this transformation explicit and take

$$SWF^\alpha = E[t^\alpha(Y)]. \tag{98}$$

Here  $t^\alpha$  is some transformation of the observable  $Y$  which might depend on a parameter  $\alpha$ . Correspondingly, define

$$g^\alpha(X) = E[t^\alpha(Y^1) - t^\alpha(Y^0)|X],$$

take  $\mathcal{G}^\alpha$  as the identified set for  $g^\alpha$ , and let  $\succeq^{\mathcal{G}^\alpha}$  be the implied preference over policies. A particularly tractable class of transformations are the quadratic ones,<sup>12</sup>

$$t^\alpha(y) = \alpha^0 + \alpha^1 y + \alpha^2 y^2.$$

In order to keep the support of  $t^\alpha(y)$  in the interval  $[0, 1]$ , we can normalize w.l.o.g. to get

$$t^\alpha(y) = y + \alpha(y - y^2), \tag{99}$$

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<sup>12</sup>The notation gets slightly confusing here: All superscripts are indices, except for superscript 2 over  $Y$ , which is an exponent.

where  $-1 \leq \alpha \leq 1$ , and the planner's taste for redistribution is increasing in  $\alpha$ . This class of transformations gives

$$\begin{aligned} g^\alpha(X) &= E[t^\alpha(Y^1)|X] - E[t^\alpha(Y^0)|X] \\ &= (1 + \alpha)E[Y^1 - Y^0|X] - \alpha E[Y^{1^2} - Y^{0^2}|X] \\ &= g^0(X) + \alpha \left( g^0(X) - E[Y^{1^2} - Y^{0^2}|X] \right). \end{aligned} \quad (100)$$

Put differently, all  $g^\alpha$  are on a line in the space  $\mathbb{R}^X$ , where the line points in direction of the function

$$\tilde{g}(X) := g^0(X) - E[Y^{1^2} - Y^{0^2}|X].$$

What now about the identified set  $\mathcal{G}^\alpha$ ? Any answer must depend on the particular source of identification, we shall discuss instrumental variables as reviewed in appendix A.1. In immediate generalization of the results reviewed there, the upper and lower bounds of the rectangular set  $\mathcal{G}^\alpha$  are given by

$$\begin{aligned} \bar{g}^\alpha(X) &= g^{mm,\alpha}(X) + E[1 - D|X, Z = z^1] \\ \underline{g}^\alpha(X) &= g^{mm,\alpha}(X) - E[D|X, Z = z^0], \end{aligned} \quad (101)$$

where

$$\begin{aligned} g^{mm,\alpha}(X) &= E[(Y + \alpha(Y - Y^2))D|X, Z = z^1] \\ &\quad - E[(Y + \alpha(Y - Y^2))(1 - D)|X, Z = z^0] \\ &= g^{mm,0}(X) + \alpha g^{mm,0}(X) \\ &\quad - \alpha (E[Y^2 D|X, Z = z^1] - E[Y^2(1 - D)|X, Z = z^0]). \end{aligned} \quad (102)$$

This implies that all  $\mathcal{G}^\alpha$  are just translations of  $\mathcal{G}^0$  along a line in direction of the function

$$\tilde{g}(X) := g^{mm,0}(X) - (E[Y^2 D|X, Z = z^1] - E[Y^2(1 - D)|X, Z = z^0]).$$

## Appendix D: Proofs

### Section 2

#### Proof of Lemma 1 :

The first claim is immediate since either  $h^a \succ^{\mathcal{G}} h^b$  or  $h^b \succ^{\mathcal{G}} h^a$  holds if and only if  $h^{ab} \in \widehat{\mathcal{G}} \cup \mathcal{G}^*$ .

Next, note that either  $h^a \succ^{\mathcal{G}} h^b$  or  $h^b \succ^{\mathcal{G}} h^a$  (i.e., a strict ordering of  $h^a, h^b$ ) holds if and only if  $h^{ab} \notin \left( \bigcup_{g \in \mathcal{G}} g^\perp \right)$ : Since  $\mathcal{G}$  is convex and therefore connected, and since the inner product is continuous, the set  $\langle h^{ab}, \mathcal{G} \rangle$  is connected. It therefore contains both positive

and non-positive values only if it contains 0. This is the case if there is a  $g \in \mathcal{G}$  such that  $\langle h^{ab}, g \rangle = 0$ , i.e., if there is a  $g \in \mathcal{G}$  such that  $h^{ab} \in g^\perp$ .

The second claim then follows from the fact that the set of  $h^a, h^b$  for which either  $h^a \succeq^{\mathcal{G}} h^b$  or  $h^b \succeq^{\mathcal{G}} h^a$  holds is the the closure of the set for which either  $h^a \succ^{\mathcal{G}} h^b$  or  $h^b \succ^{\mathcal{G}} h^a$  holds: Let  $g^1 = \operatorname{argmin}_{g \in \mathcal{G}} \langle h^{ab}, g \rangle$ . If  $h_n$  is a sequence in  $d\mathcal{H} \cap \{h : \min_{g \in \mathcal{G}} \langle h, g \rangle > 0\}$  converging to  $h^{ab}$ , then

$$\min_{g \in \mathcal{G}} \langle h^{ab}, g \rangle = \langle h^{ab}, g^1 \rangle = \lim_n \langle h_n, g^1 \rangle \geq \liminf_n \min_{g \in \mathcal{G}} \langle h_n, g \rangle \geq 0.$$

Conversely, for any  $h^{ab}$  such that  $\min_{g \in \mathcal{G}} \langle h^{ab}, g \rangle \geq 0$  there is an  $h$  in any of the neighborhoods of  $h^{ab}$  such that  $\min_{g \in \mathcal{G}} \langle h, g \rangle > 0$ . We can choose for instance  $h = h^{ab} + \epsilon h'$ , where  $h'$  is such that  $\min_{g \in \mathcal{G}} \langle h', g \rangle > 0$ . Such a hyperplane separating 0 from  $\mathcal{G}$  exists: Choose  $h' = \operatorname{argmin}_{g \in \overline{\mathcal{G}}} \|g\|$ . By  $0 \notin \overline{\mathcal{G}}$  we have  $\|h'\| > 0$ ,  $\langle h', g \rangle > 0$  for all  $g \in \overline{\mathcal{G}}$  follows from convexity of  $\mathcal{G}$ .  $\square$

### Proof of Lemma 2

1. For  $\alpha < 1$ , this follows immediately from the definition of independence if we take  $h^c = 0$ . For  $\alpha > 1$ , this follows from reversing the role of  $h^a, h^b$  and  $\alpha h^a, \alpha h^b$ .
2. By independence,  $h^a \succ h^b$  if and only if  $.5h^a + .5h^c \succ .5h^b + .5h^c$ . Now apply claim (1), using  $\alpha = 2$ .
3. This follows immediately from claim (2).
4. That this set is a cone follows from (1). That it is convex follows by applying independence twice.

$\square$

### Proof of Lemma 4 :

Suppose to the contrary that there are three vectors  $h^1, h^2, h^3 \in \mathcal{H}'$ , such that  $h^1 - h^3$  and  $h^2 - h^3$  are linearly independent. Take any  $g$  from the interior of  $\mathcal{G}$ . Define  $\alpha^i = \langle h^i - h^3, g \rangle$  for  $i = 1, 2$ . Since  $\mathcal{H}'$  is totally ordered and  $g$  is in the interior of  $\mathcal{G}$ ,  $\alpha_i \neq 0$  for  $i = 1, 2$ . Let

$$h^* = (h^1 - h^3) - \frac{\alpha^1}{\alpha^2} (h^2 - h^3).$$

$h^*$  lies in the linear space spanned by  $h^1 - h^3, h^2 - h^3$ . Furthermore  $\langle h^*, g \rangle = 0$ . We can find two policy vectors  $h^4, h^5$  in the triangle spanned by  $h^1, h^2, h^3$ , and hence in the policy set  $\mathcal{H}'$ , such that  $h^4 - h^5 = \beta h^*$  for some  $\beta \neq 0$ . Since  $g$  is in the interior of  $\mathcal{G}$ , we can now find

$g^1, g^2 \in \mathcal{G}$ , such that  $\langle h^4 - h^5, g^1 \rangle < 0 < \langle h^4 - h^5, g^2 \rangle$ . Contradiction.  $\square$

**Proof of Lemma 5 :**

First suppose  $\lambda c$  lies in the interior of  $\mathcal{G}$ . By definition of  $\mathcal{H}'$ ,  $\langle h^a - h^b, g \rangle = 0$  for all  $h^a, h^b \in \mathcal{H}'$  and  $g = \lambda c$ . Since  $g$  lies in the interior of  $\mathcal{G}$ , the sign of  $\langle h^a - h^b, g' \rangle$  is therefore positive and negative for some  $g' \in \mathcal{G}$ .

Suppose on the other hand that  $\lambda c$  does not lie in  $\mathcal{G}^o$  for any  $\lambda$ . By the Hahn-Banach separating hyperplane theorem, applied to the sets  $\mathcal{G}$  and  $\{\lambda c : \lambda \in \mathbb{R}\}$  there exists an  $\tilde{h}$  such that

$$\sup_{\lambda \in \mathbb{R}} \langle \tilde{h}, \lambda c \rangle \leq \inf_{g \in \mathcal{G}} \langle \tilde{h}, g \rangle.$$

The left hand side is bounded only if  $\langle \tilde{h}, c \rangle = 0$ , which thus must hold, and therefore

$$0 \leq \inf_{g \in \mathcal{G}} \langle \tilde{h}, g \rangle.$$

Therefore any policy pair  $h^a, h^b \in \mathcal{H}'$  such that  $h^{ab}$  is proportional to  $\tilde{h}$  is weakly ordered.  $\square$

### Section 3

**Proof of Lemma 6 :**

That  $g^\phi \leq \bar{g}^\phi$ , as defined in the statement of the lemma, follows immediately from  $g^\phi = \frac{\partial \phi}{\partial F} \cdot g^y$  and the decomposition of  $g^y$  in equation (44); similarly for  $\underline{g}^\phi$  and for the panel data case.

To see that the IV bounds are tight, we have to again invoke monotonicity of the first stage relationship mapping  $Z$  to  $D$ , as in appendix A.1. Under these monotonicity assumptions, the same arguments as in appendix A.1, equations (72) go through, with the appropriate substitutions made.  $\square$

**Proof of Corollary 1 :**

The first two claims are immediate from lemma 6. For the last claim, we have to show that in all neighborhoods of  $h^0$  there exists an  $h$  such that  $\phi(\langle h, g^y \rangle) > \phi(\langle h^0, g^y \rangle)$  if and only if there exists an  $h$  such that  $\langle h - h^0, g^\phi \rangle > 0$ . This follows from the definition of Fréchet differentiability, which implies that the signs of either expression are equal for  $h$  close enough to  $h^0$ , as long as either expression is different from 0.  $\square$

### Section 4

**Proof of Lemma 7 :**

Since  $\mathcal{H}'$  is unconstrained, regret  $R(g, g')$  for any  $g, g'$  is given by

$$R(g, g') = E [\mathbf{1}(\text{sign}(g) \neq \text{sign}(g')) \cdot |g|].$$

In particular, choosing

$$g = \begin{cases} \bar{g} & \text{if } |\bar{g}| > |\underline{g}| \\ \underline{g} & \text{else} \end{cases}$$

and

$$g' = \begin{cases} \underline{g} & \text{if } |\bar{g}| > |\underline{g}| \\ \bar{g} & \text{else} \end{cases}$$

gives  $R(g, g') = E [\mathbf{1}(\bar{g} > 0 > \underline{g}) \cdot \max(\bar{g}, -\underline{g})]$ . It is immediate that this is also the upper bound on regret.

Inequality (56) then follows from

$$|\mathbf{1}(\bar{g} > 0 > \underline{g}) \cdot \max(\bar{g}, -\underline{g})| < w$$

for all  $X$ . To see the final claim, choose  $\underline{g} = \max(-\delta, -w/2)$  and  $\bar{g} = \underline{g} + w$ .  $\square$

**Proof of Lemma 8 :**

First, note that

$$\langle |h^*(g) - h^*(g')|, c \rangle \leq \langle |h^*(g)| + |h^*(g')|, c \rangle \leq 2C \quad (103)$$

due to the budget constraint.

Next, let  $\tilde{h} = h^*(g) - h^*(g')$ . Note that regret is invariant to adding or subtracting a multiple of the function  $c$  to all elements of  $\mathcal{G}$ , since for all admissible  $\tilde{h}$  we have  $\langle \tilde{h}, c \rangle = 0$ . By equation (52), we can thus assume w.l.o.g. that  $h^*(g') = \mathbf{1}(g' > 0) + \kappa' \mathbf{1}(g' = 0)$ . Since  $-h^*(g') \leq \tilde{h} \leq 1 - h^*(g')$ , it follows that  $\tilde{h} > 0$  only if  $\underline{g} \leq g' \leq 0$  and thus  $\bar{g} = \underline{g} + w \leq w$ . Similarly  $\tilde{h} < 0$  only if  $\bar{g} \geq g' \geq 0$  and thus  $\underline{g} = \bar{g} - w \geq -w$ . We then have

$$\begin{aligned} R(g, g') &= \langle \tilde{h}, g \rangle \\ &\leq \langle \tilde{h}, \bar{g} \cdot \mathbf{1}(\tilde{h} \geq 0) + \underline{g} \cdot \mathbf{1}(\tilde{h} < 0) \rangle \\ &\leq \langle \tilde{h}, w \cdot \mathbf{1}(\tilde{h} \geq 0) - w \cdot \mathbf{1}(\tilde{h} < 0) \rangle \\ &= \langle |\tilde{h}|, w \rangle \end{aligned} \quad (104)$$

The bound (57) now follows, since the right hand side of inequality (104) is maximized, subject to the constraint (103) ( $\langle |\tilde{h}|, c \rangle \leq 2C$ ), and subject to  $|\tilde{h}| \leq 1$ , by concentrating  $|\tilde{h}|$  on the set where  $w/c$  is largest, yielding the first claim.

To see that bound (57) is tight, construct a set  $\mathcal{G}$  as follows: Suppose w.l.o.g.  $\delta < w$ . Choose

$$g = \begin{cases} w - \delta & \text{if } w/c > \omega' \\ -w + \delta & \text{else} \end{cases}$$

where  $\omega'$  is such that  $E[c \cdot \mathbf{1}(w/c \geq \omega')] = C$ . Choose

$$g' = \begin{cases} \delta & \text{if } \omega' > w/c > \omega \\ -\delta & \text{else} \end{cases}$$

Then  $h^*(g) = \mathbf{1}(w/c \geq \omega')$ ,  $h^*(g') = \mathbf{1}(\omega' > w/c > \omega)$ ,  $\tilde{h} \cdot g = (w - \delta)\mathbf{1}(w/c \geq \omega') - (-w + \delta)\mathbf{1}(\omega' > w/c > \omega)$ , and thus

$$R(g, g') = E[(w - \delta)\mathbf{1}(w/c \geq \omega')].$$

Specializing these results to the constant  $c$  case is immediate.  $\square$

### Proof of Lemma 9 :

Since the policy set is unconstrained and the identified set is rectangular, the minimax regret strategy solves, for each  $X$ , the problem

$$\begin{aligned} h^{mmr}(X) &= \operatorname{argmin}_{z \in [0,1]} \max_{y \in [\underline{g}(X), \bar{g}(X)]} (j(y) - z \cdot y) \\ &= \operatorname{argmin}_{z \in [0,1]} \max_{y \in \{\underline{g}(X), \bar{g}(X)\}} ((\mathbf{1}(y > 0) - z) \cdot y). \end{aligned} \quad (105)$$

If  $\underline{g}(X) < 0 < \bar{g}(X)$ , this is maximized in  $z$  by setting

$$-\underline{g}(X) \cdot z = (1 - z) \cdot \bar{g}(X)$$

which gives

$$z = \frac{\bar{g}(X)}{\bar{g}(X) - \underline{g}(X)}.$$

If  $\underline{g}(X) > 0$  or  $\bar{g}(X) < 0$ , it is immediate that the optimal  $z$  equals 1 or 0, which shows the first claim. We get, furthermore, that

$$\min_{z \in [0,1]} \max_{y \in [\underline{g}(X), \bar{g}(X)]} (j(y) - z \cdot y) = \mathbf{1}(\underline{g}(X) < 0 < \bar{g}(X)) \cdot \frac{-\bar{g}(X) \cdot \underline{g}(X)}{\bar{g}(X) - \underline{g}(X)},$$

which gives the second claim. Finally, this expression for minimax regret is maximized among sets  $\mathcal{G}$  of width  $w$  by centering  $\mathcal{G}$  at 0, which shows the third claim.  $\square$

**Proof of Lemma 10 :**

Denote  $\hat{g}(\mathcal{P}) = E[g|\mathcal{P}]$ , similar for  $\hat{g}(\mathcal{O})$ . By iterated expectations,  $\hat{g}(\mathcal{P}) = E[\hat{g}(\mathcal{O})|\mathcal{P}]$ . We get

$$\begin{aligned} ER^{\mathcal{P}} - ER^{\mathcal{O}} &= E[V(\hat{g}(\mathcal{O}))] - E[V(\hat{g}(\mathcal{P}))] \\ &= E[E[V(\hat{g}(\mathcal{O}))|\mathcal{P}] - V(E[\hat{g}(\mathcal{O})|\mathcal{P}])]. \end{aligned}$$

Convexity of  $V$  and Jensen's inequality imply

$$E[V(\hat{g}(\mathcal{O}))|\mathcal{P}] \geq V(E[\hat{g}(\mathcal{O})|\mathcal{P}]).$$

for every  $\mathcal{P}$ , which proves the claim.  $\square$

**Proof of Lemma 11 :**

Suppose first  $\bar{g} > \tilde{g} > 0 > \hat{g}$  for all  $X$ , where  $\tilde{g} := \hat{g} + (1/2 - \gamma)w$  and  $\bar{g} = \hat{g} + 1/2w$ . Then  $j(-g(X) \cdot \text{sign}(\hat{g}(X))) = g(X)\mathbf{1}(g(X) > 0)$ . Recall from equation (67) that

$$ER = E_g[E_X[j(-g(X) \cdot \text{sign}(\hat{g}(X)))]|\mathcal{P}].$$

Expected regret is thus given by the expectation over  $X$  of the following expression:

$$\begin{aligned} & \frac{1}{w^2\gamma(1-\gamma)} \left( \int_0^{\bar{g}} (\bar{g} - y)ydy - \int_0^{\tilde{g}} (\tilde{g} - y)ydy \right) \\ &= \frac{1}{w^2\gamma(1-\gamma)} \left( (\bar{g}^3/2 - \bar{g}^3/3) - (\tilde{g}^3/2 - \tilde{g}^3/3) \right) \\ &= \frac{1}{6w^2\gamma(1-\gamma)} (\bar{g}^3 - \tilde{g}^3). \end{aligned} \tag{106}$$

This proves the claim for the case  $\bar{g} > \tilde{g} > 0 > \hat{g}$ . Similar calculations for the cases  $\bar{g} > 0 > \tilde{g} > \hat{g}$ ,  $\bar{g} < 0 < \tilde{g} < \hat{g}$  and  $\bar{g} < \tilde{g} < 0 < \hat{g}$  show that it holds in general.  $\square$

## Appendix

**Proof of Lemma 12:**

Assume w.l.o.g. that  $g^{mm} = 0$  and  $E[1 - D|X, Z = z^1] = 1$ , and let  $b = E[D|X, Z = z^0]$ . We get

$$g = u^1 - bu^2,$$

and thus  $u^1 = g + bu^2$ . The posterior density of  $g(x)$  is then proportional to

$$p(g(X) = y | \mathcal{P}) = \text{const.} \cdot \int \mathbf{1}(0 \leq u^2(x) \leq 1, 0 \leq y + bu^2(x) \leq 1) du^2(x)$$

Consider first the case  $b < 1$ . Then  $\gamma = \frac{b}{1+b}$  and

$$p(g(X) = y | \mathcal{P}) = \text{const.} \cdot \begin{cases} \frac{y+b}{b} & \text{for } y \in [-b, 0] \\ 1 & \text{for } y \in [0, 1-b] \\ \frac{1-y}{b} & \text{for } y \in [1-b, 1]. \end{cases}$$

Similar expressions hold in the case  $b \geq 1$ . The statement of Lemma 12 then follows by rescaling and normalization.  $\square$

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