

# Some unpleasant properties of log-linearized solutions when the nominal rate is zero.\*†

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## Abstract

A growing body of recent research examines the nonlinearity created by a zero lower bound on the nominal interest rate. It is common practice to log-linearize the other equilibrium restrictions around a deterministic steady state with a stable price level. This paper shows that the resulting log-linearized solutions have some unpleasant properties. We make this point using a tractable New Keynesian model. The properties of log-linearized and exact solutions are quite different. The slope of both the AD and AS schedules in the log-linearized solution differs from that of the exact solution for a broad range of parameters. One implication of this result is that the log-linearized solution produces an incorrect sign for the response of hours to a tax cut.

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# 1 Introduction

The recent experiences of Japan, the United States, and Europe with zero/near-zero nominal interest rates have raised new questions about the conduct of monetary and fiscal policy in a liquidity trap. These events have produced a large and growing body of new research that explicitly models the zero bound on the nominal interest rate. Findings from this literature have already influenced the thinking and actions of monetary policy makers. One strand of this recent literature analyzes the effectiveness of monetary and/or fiscal policy in New Keynesian models under the assumption that the monetary authority pursues a Taylor rule and that the monetary policy feedback rule has a nonlinearity due to the zero lower bound on the nominal interest rate. Some recent examples include [Del Negro, Eggertsson, Ferrero, and Kiyotaki \(2010\)](#); [Bodenstein, Erceg, and Guerrieri \(2009\)](#); [Eggertsson and Krugman \(2010\)](#); [Eggertsson \(2011\)](#); [Woodford \(2011\)](#); [Christiano, Eichenbaum, and Rebelo \(2011\)](#) and [Erceg and Lindé \(2010\)](#). All of these papers explicitly model the non-differentiability created by a zero bound on the nominal interest rate on the Taylor rule. However, all of these papers use log-linearized versions of the remaining equilibrium conditions when solving the model.

A second strand of this recent literature analyzes optimal monetary policy in the presence of a zero bound on nominal interest rate. Examples include [Adam and Billi \(2006\)](#) and [Nakov \(2008\)](#). The last two papers use global methods to derive the optimal monetary policy. However, the only nonlinearity in the specifications considered in these papers is the implementability restriction that the nominal interest rate be non-negative. The remaining implementability conditions are log-linearized.

One conclusion that has emerged from models solved in this way is that the dynamics of the New Keynesian model are very different when the nominal interest rate is zero. [Eggertsson \(2011\)](#) finds that output falls when the labor tax is lowered when the nominal interest rate is zero. [Braun and Waki \(2006\)](#) and [Christiano, Eichenbaum, and Rebelo \(2011\)](#) find that the output response to a positive technology shock is negative when the nominal interest rate is zero and [Christiano,](#)

Eichenbaum, and Rebelo (2011); Woodford (2011); and Erceg and Lindé (2010) find that the size of the government purchase multiplier is substantially larger when the nominal interest rate is zero.

To understand the nature of the problem it is instructive to consider a specific example. Christiano, Eichenbaum, and Rebelo (2011) (CER), investigate the size of the government purchase multiplier when the nominal interest rate is zero in the following way. They first parameterize a New Keynesian model in a way that implies that the steady state inflation rate is zero and then use log-linearized equations for all of the equilibrium conditions except that Taylor rule. They then drive the interest rate to fall to zero by shocking the preference discount factor. Finally, they compute the government purchase multiplier by comparing the outcome from this impulse to an alternative scenario where government purchases impulsed at the same time as the preference discount rate.

It takes a big (5 percent) shock in the preference discount rate to induce a binding zero interest rate. This is not specific to the parameterization considered by CER. Coenen, Orphanides, and Wieland (2004), for instance, estimate a New Keynesian model on U.S. data from 1980 to 1999. They find that the probability of a shock driving the nominal interest rate to zero is very low, when the long-run inflation target rate is 2%. Only very large shocks produce a binding zero nominal interest rate in their estimated specification.

It is well known that log-linear solution methods and perturbation methods more generally only work well within a given radius of the point at which the approximation that is taken and that outside of this radius these solutions break down (See e.g. Den Haan and Rendahl (2009) and Aldrich and Kung (2009)).

The objective of this paper is to provide evidence that such a breakdown occurs when one analyzes the zero bound on nominal interest rates using log-linear solution techniques.

We wish to emphasize at the outset that the problem we are raising is *not* about how the lower bound on the interest rate is handled. Instead it pertains to how the *remaining* equilibrium

conditions are specified. In this literature the convention is to log-linearize them.<sup>1</sup>

We consider an New Keynesian (NK) model with a zero bound constraint that is similar to specifications considered in [Eggertsson and Woodford \(2003\)](#) and [Eggertsson \(2011\)](#). One important distinction is that we assume [Rotemberg \(1996\)](#) price adjustment costs. This choice is very convenient. The log-linearized equilibrium conditions for our model are identical to those considered by e.g. [Eggertsson \(2011\)](#) with a suitable choice of the price adjustment cost parameter. Moreover, with [Rotemberg \(1996\)](#) price adjustment costs the exact nonlinear dynamics of the model in a liquidity trap can be represented with two equations in the two variables labor input and inflation. These two equations can be interpreted as determining aggregate demand (AD) and aggregate supply (AS). This makes it possible to produce a partial analytical characterization of the properties of the “exact” solution as well as the log-linearized solution.

One advantage of our setup is that we can compute the exact nonlinear solution that explicitly recognizes these costs. Previous work by [Eggertsson \(2011\)](#) and [Eggertsson and Krugman \(2010\)](#) has argued that the AD and AS schedules are both upward sloping and that the AD schedule cuts the AS schedule from below when the economy is in a liquidity trap. An implication of this result is that a labor tax cut lowers labor input and thus output (AS shifts right). These results are derived using a log-linearized solution. The log-linearized solution to our model exhibits these same properties. The fact that the nonlinear equilibrium conditions can be expressed also be expressed as (nonlinear) AS and AD schedules makes it possible to directly compare the properties of AS and AD across the exact and log-linearized solutions.<sup>2</sup> We find that the AD and AS schedules for the “exact” solution have four distinct configurations depending on the size of the shock and the degree of price rigidity. When the model is calibrated to reproduce outcomes from the U.S. economy the “exact” solution has the property that a tax cut increases labor input instead.

The dynamics of the “exact” solution are different from the log-linearized solution in another respect. Equilibrium is unique for the log-linearized model. Equilibrium is not always unique in the “true” model. Depending on the parameterization there can be either one and two equilibria.

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<sup>1</sup>[Wolman \(2005\)](#) is one notable exception to this common practice.

<sup>2</sup>This is no longer possible under Calvo price setting.

Our research is most closely related to work by [Braun and Waki \(2010\)](#) who investigate the size of the government purchase multiplier in a NK model with capital accumulation in a liquidity trap. They consider both [Calvo \(1983\)](#) and Rotemberg price setting schemes. The log-linearized solution exhibits large approximation errors under either form of price adjustment. They find that the implied resource costs of price dispersion/price adjustment are as large as 16% of output using the log-linearized solution for a 5% shock to the preference discount factor. Recognizing these resource costs using a global solution method reduces the size of the government purchase multiplier by as much as 50%.

Our research is also related to work by [Braun and Körber \(2011\)](#). They calibrate a NK model with capital formation to Japanese data and confront the model with Japan's experience of zero interest rates. In contrast to the previous papers cited above they find that output increases in response to labor tax cuts and improvements in technology when they solve the model using a global solution method.

The remainder of the paper proceeds as follows. In [Section 2](#) we describe the model. In [Section 3](#) we derive some analytic results about the slopes of the AD and AS schedules in a liquidity trap. We also use numerical simulations to provide a more complete characterization of the slopes of the AD and AS schedules. [Section 4](#) considers the response of labor input, GDP and inflation to a labor tax cut. In [section 5](#) we summarize some robustness checks we have performed and [6](#) contains our concluding remarks.

## 2 The model

Consider a representative household who chooses consumption  $c_t$ , labor  $h_t$  and bond holdings  $b_t$  to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \Pi_{j=0}^t d_j \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{h_t^{1+\nu}}{1+\nu} \right\} \quad (1)$$

subject to

$$b_t + c_t = \frac{b_{t-1}(1 + R_{t-1})}{1 + \pi_t} + w_t h_t (1 - \tau_{w,t}) \quad (2)$$

where  $\nu$  governs the elasticity of labor supply and  $\sigma$  is the curvature parameter for consumption. The variable  $d_t$  is a shock to the preference discount factor.

The optimality conditions for the household's problem imply that consumption and labor supply choices satisfy

$$c_t^\sigma h_t^\nu = w_t(1 - \tau_{w,t}) \quad (3)$$

$$1 = \beta E_t \left\{ \frac{d_{t+1}(1 + R_t)}{1 + \pi_{t+1}} \left( \frac{c_t}{c_{t+1}} \right)^\sigma \right\} \quad (4)$$

Perfectly competitive final good firms use a continuum of intermediate goods  $i \in 0, 1$  to produce a single final good that can be used for consumption and investment. The final good is produced using the following production technology

$$y_t = \left( \int_0^1 y_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \quad (5)$$

The profit maximizing input demands of the final good firm are

$$y_t(i)^d = \left( \frac{p_t(i)}{P_t} \right)^{-\theta} y_t \quad (6)$$

where  $p_t(i)$  denotes the price of the good produced by firm  $i$ . The price of the final good  $P_t$  is given by

$$P_t = \left( \int_0^1 p_t(i)^{1-\theta} di \right)^{1/(1-\theta)} \quad (7)$$

Intermediate goods producers use labor to produce output:  $y_t(i) = h_t(i)$ . This production function implies that for all firms their real marginal cost  $\chi_t$  is equal to the real wage  $w_t$ :

$$w_t = \chi_t. \quad (8)$$

Producer  $i$  sets prices to maximize

$$\sum_{t=0}^{\infty} \beta^t \prod_{j=0}^t d_j \lambda_{c,t} \left[ (1 + \tau_s) p_t(i) y_t(i) - P_t \chi_t y_t(i) - \frac{\gamma}{2} (\pi_t - \pi)^2 P_t y_t \right] / P_t \quad (9)$$

where  $\tau_s$  is a subsidy to intermediate good producers. We model costly price adjustment using [Rotemberg \(1996\)](#) quadratic price adjustment costs. Calvo price adjustment is more common in the literature. When using log-linearized solution methods centered at a stable price level this choice is innocuous. By a suitable choice of parameters Calvo and Rotemberg quadratic adjustment cost specifications deliver the identical log-linearized decision rules. This equivalence does not apply more generally. We choose to use quadratic adjustment costs because one can analyze global dynamics of the nonlinear model using a two equation system that can be interpreted as AD and AS schedules. This simple structure makes it possible to derive some analytical results and to also provide insights into the nature of the breakdown of the log-linearized solution. Under Calvo pricing, variation in the dispersion of prices complicates the dynamics of the model and it is no longer possible to get analytic results.

The first order condition for this problem in a symmetric equilibrium is:

$$0 = \theta \chi_t + (1 + \tau_s)(1 - \theta) - \gamma(\pi_t - \pi)(1 + \pi_t) + \beta E_t \left\{ d_{t+1} \left( \frac{c_t}{c_{t+1}} \right)^\sigma \frac{y_{t+1}}{y_t} \gamma (\pi_{t+1} - \pi)(1 + \pi_{t+1}) \right\} \quad (10)$$

Monetary policy follows a Taylor rule that respects the zero lower bound on the nominal interest rate

$$R_t = \max(0, r_t^e + \phi_\pi \pi_t + \phi_y \hat{y}_t) \quad (11)$$

$$r_t^e = -\log(\beta) - E_t \ln(d_{t+1})$$

where  $\hat{y}_t$  denotes the log deviation of output from its steady state value. The aggregate resource constraint is given by

$$c_t = (1 - \kappa_t - \eta_t) y_t \quad (12)$$

where  $\kappa_t \equiv \frac{\gamma}{2}(\pi_t - \pi)^2$  is the resource cost of price adjustment and government purchases  $g_t = \eta_t y_t$ .<sup>3</sup> It follows that gross domestic product in our economy,  $gdp_t$ , is given by:

$$gdp_t \equiv (1 - \kappa_t)y_t. \quad (13)$$

This definition assumes that the price adjustment costs are treated as intermediate inputs and thus are subtracted when calculating GDP.

We next turn to describe how we derive AS and AD schedules that describe the evolution of  $\pi_t$  and  $h_t$ . To obtain the aggregate supply (AS) schedule, we first combine the labor demand decision of the firm, (8), and the labor supply decision of the household, (3), to express real marginal costs  $\chi_t$  as

$$\chi_t = \frac{c_t^\sigma h_t^\nu}{(1 - \tau_{w,t})} \quad (14)$$

Next we use expressions (12) and (14) to substitute marginal costs and consumption out of the firm's optimal price setting restriction (10) and obtain the AS schedule

$$0 = \frac{\theta(1 - \kappa_t - \eta_t)^\sigma}{(1 - \tau_{w,t})} h_t^{\nu+\sigma} + (1 + \tau_s)(1 - \theta) - \Psi'_t(1 + \pi_t) + \beta E_t \left\{ d_{t+1} \left( \frac{1 - \kappa_{t+1} - \eta_{t+1}}{1 - \kappa_t - \eta_t} \right)^\sigma \left( \frac{h_t}{h_{t+1}} \right)^{\sigma-1} \Psi'_{t+1}(1 + \pi_{t+1}) \right\} \quad (15)$$

where  $\Psi'_t = \gamma(\pi_t - \pi)$ . The second equation is a nonlinear version of the New Keynesian IS curve, or aggregate demand curve (AD). It is obtained by substituting consumption out of the household's intertemporal Euler equation (4) using (12) and the production function. The resulting AD schedule is

$$1 = \beta E_t \left\{ \frac{d_{t+1}(1 + R_t)}{1 + \pi_{t+1}} \left( \frac{1 - \kappa_{t+1} - \eta_{t+1}}{1 - \kappa_t - \eta_t} \right)^\sigma \left( \frac{h_t}{h_{t+1}} \right)^\sigma \right\} \quad (16)$$

where  $R_t$  is given by (11).

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<sup>3</sup>We introduce new notation for  $\kappa_t$  because it allows us to isolate the role of omitting the resource costs of price adjustment from the resource constraint.

We find it convenient to express the aggregate demand and supply schedules in terms of labor input (or gross production) rather than GDP. This choice allows us to highlight the fact that the response of labor differs according to the solution method and also provide some intuition for why.

In order to analyze the effects of a zero interest rate on the dynamic properties of the model we need to describe how the nominal interest rate falls to zero.

Following [Eggertsson and Woodford \(2003\)](#) and [Eggertsson \(2011\)](#), we assume that the economy starts off in a perfect foresight steady state with the discount factor  $d = 1$  and zero inflation,  $\pi = 0$ . Steady-state hours are given by  $h = ((\theta - 1)(1 + \tau_s)(1 - \tau_w)/[\theta(1 - \eta)^\sigma])^{1/(\sigma + \nu)}$  and the steady-state value of the nominal interest rate is:  $R = 1/\beta - 1$ . At time-0, agents in the model economy realize that the preference shock  $d_t$  follows a two-state Markov chain with states  $(d^L, 1)$ , the initial condition  $d_1 = d^L$ , and the transition probabilities  $P(d_{t+1} = d^L | d_t = d^L) = p < 1$  and  $P(d_{t+1} = 1 | d_t = 1) = 1$ . We assume that the government policy  $(\tau_{w,t}, \eta_t)$  depends only on  $d_{t+1}$  and is independent of time  $t$ . The economy continues in state  $L$  until a new shock to the discount factor shifter arrives at which point  $d$  reverts to 1 and the economy returns to its perfect foresight steady state and remains there in all subsequent periods. The preference shock  $d^L$  is taken large enough to cause a binding zero lower bound when  $d_t = d^L$ .

Following the previous literature we consider an equilibrium in which allocations and prices take on one of two distinct values: one value obtains when the nominal rate is zero and the other applies when the nominal rate is positive and at its steady-state. We will use the superscript  $L$  to denote the former value and no subscript to indicate the latter value.

The equilibrium in state  $L$  (more specifically  $(h^L, \pi^L)$ ) is characterized by the following expressions for the AS and AD schedules

$$0 = \theta \frac{(1 - \kappa^L - \eta^L)^\sigma (h^L)^{\sigma + \nu}}{(1 - \tau_w)} + (1 + \tau_s)(1 - \theta) - \gamma \pi^L (1 + \pi^L) + p \beta d^L \gamma \pi^L (1 + \pi^L) \quad (17)$$

$$1 = p \left( \frac{\beta d^L}{1 + \pi^L} \right) + (1 - p) \left( \frac{(1 - \kappa^L - \eta^L)^\sigma (h^L)^\sigma}{(1 - \eta)^\sigma h^\sigma} \right) \quad (18)$$

where  $\kappa^L = \frac{\gamma}{2}(\pi^L)^2$ .<sup>4</sup>

To highlight the role of the resource costs of price adjustment we consider the following *mis-specified nonlinear* AD and AS schedules that treat  $\kappa^L$  as invariant at zero:

$$0 = \theta \frac{(1 - \eta^L)^\sigma (h^L)^{\sigma + \nu}}{(1 - \tau_w)} + (1 + \tau_s)(1 - \theta) - \gamma \pi^L (1 + \pi^L) + p \beta d^L \gamma \pi^L (1 + \pi^L) \quad (19)$$

$$1 = p \left( \frac{\beta d^L}{1 + \pi^L} \right) + (1 - p) \left( \frac{(1 - \eta^L)^\sigma (h^L)^\sigma}{(1 - \eta)^\sigma h^\sigma} \right) \quad (20)$$

### 3 Properties of the AD and AS schedules in a liquidity trap.

We turn now to analyze the properties of the AD and AS schedules in the low state with  $R = 0$ . We provide two sets of results. Some results are derived analytically. Other results are produced by simulating the model. In what follows we are assuming that it is a shock to  $d_t$  that brings the nominal interest rate to zero.

#### 3.1 Analytical results

Proposition 1 establishes a condition under which the AD schedule defined in this way is downward sloping.

**Proposition 1.** *The AD schedule is downward sloping at  $(h^L, \pi^L)$  if*

$$\frac{\sigma (h^L)^\sigma (1 - \kappa^L - \eta)^{\sigma - 1} (\kappa^L)'}{(1 - \eta)^\sigma h^\sigma} < \frac{p \beta d^L}{(p - 1)(1 + \pi^L)^2} \quad (21)$$

and  $\pi^L < 0$  where  $(\kappa^L)' = \gamma \pi^L$ .

#### Proof

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<sup>4</sup>Other equilibrium objects are recovered as  $y^L = h^L$ ,  $gdp^L = (1 - \kappa^L)y^L$ ,  $c^L = (1 - \kappa^L - \eta^L)y^L$ , etc. Strictly speaking, not all pairs  $(h^L, \pi^L)$  that solve this system are equilibria:  $(1 - \kappa^L - \eta^L) \geq 0$  and  $R + \phi_\pi \pi^L + \phi_y \hat{y}^L - \log d^L \leq 0$  have to be satisfied.

The slope of the aggregate demand schedule (18) is given by

$$\frac{D\pi^L}{Dh^L} = -\frac{AD_{h^L}}{AD_{\pi^L}} \equiv -\frac{\partial AD(h^L, \pi^L)/\partial h^L}{\partial AD(h^L, \pi^L)/\partial \pi^L} = \frac{\frac{\sigma(1-p)(1-\kappa^L-\eta)^\sigma (h^L)^{\sigma-1}}{(1-\eta)^\sigma h^\sigma}}{\frac{p\beta d^L}{(1+\pi^L)^2} + \frac{(1-p)\sigma(h^L)^\sigma (1-\kappa^L-\eta)^{\sigma-1} (\kappa^L)'}{(1-\eta)^\sigma h^\sigma}} \quad (22)$$

Consider the numerator first. It is unambiguously positive. Next consider the denominator. It is negative  $\iff$  (3.1) is satisfied and  $\pi^L < 0$ .

□ Proposition 1 has several immediate implications. Consider first the slope of the misspecified AD schedule.

**Lemma 1.** *The misspecified nonlinear AD schedule is upward sloping*

This is clear since the misspecified nonlinear AD schedule (incorrectly) imposes  $\kappa^L = (\kappa^L)' = 0$ . These restrictions on  $\kappa^L$  are of interest because this is what happens if one log-linearizes the equilibrium conditions around a steady-state with zero inflation. Since the steady-state resource costs are zero, the values of  $\kappa^L$  and  $(\kappa^L)'$  in the log-linearized system are both zero.

When  $p = 0$ , inspection of (22) reveals that the misspecified nonlinear AD schedule is vertical. This is identical to a previous finding that Eggertsson (2011) derives using a log-linearized system. When  $p$  exceeds zero the sign of equation (22) is positive for all  $\{p : 0 < p < 1\}$ . This result also echoes the finding of Eggertsson (2011) that the log-linearized AD schedule is unambiguously *upward* sloping for all  $\{p : 0 < p < 1\}$  when the nominal interest rate is zero.

The resource costs of price adjustment don't appear in the log-linearized system. But they do appear in the true nonlinear model. How do these resource costs affect the slope of the AD schedule? Recall that  $\kappa^L = \gamma/2(\pi^L)^2 > 0$  and  $(\kappa^L)' = \gamma\pi^L < 0$  in the true model with quadratic price adjustment costs. Inspection of (3.1) reveals that these costs have a big impact on the slope of AD schedule when  $p$ , which governs the expected duration of the low state is small. If  $p = 0$  the AD schedule in the nonlinear model is downward sloping. By continuity the AD schedule is also downward sloping for  $p > 0$  but sufficiently small. What about for larger values of  $p$ ? It is hard to ascertain what happens from (3.1). In addition to the size of  $p$ , the size of the shock  $d^L$

and the configuration of other parameters also matter.

To investigate the signs of the slopes of the "exact" solution for larger values of  $p$ , we will carry out numerical experiments using parameter values and shocks that have been used elsewhere in the literature. Before discussing the numerical results we first present some analytical results on the properties of the AS schedule.

**Proposition 2** *The AS schedule is upward sloping at  $(h^L, \pi^L)$  if*

$$\theta\sigma(1 - \kappa^L - \eta)^{\sigma-1}(\kappa^L)'(h^L)^{\sigma+\nu} + (1 - \tau_w)\gamma(1 - p\beta d^L)(1 + 2\pi^L) > 0 \quad (23)$$

and  $\pi_L < 0$ .

**Proof** The slope of the AS schedule is given by:

$$\frac{D\pi^L}{Dh^L} = -\frac{AS_{h^L}}{AS_{\pi^L}} \equiv -\frac{\partial AS(h^L, \pi^L)/\partial h^L}{\partial AS(h^L, \pi^L)/\partial \pi^L} \quad (24)$$

$$= \frac{\theta(\sigma + \nu)(1 - \kappa^L - \eta)^\sigma (h^L)^{\sigma+\nu-1}}{\theta\sigma(1 - \kappa^L - \eta)^{\sigma-1}(\kappa^L)'(h^L)^{\sigma+\nu} + (1 - \tau_w)\gamma(1 - p\beta d^L)(1 + 2\pi^L)} \quad (25)$$

The result follows immediately from the observation that the numerator is positive.  $\square$

The slope of the AS schedule also depends on the treatment of the resource costs of price adjustment and the expected duration of zero interest rates,  $p$ , the size of the shock,  $d^L$  and the parameterization of the model. Consider first, the misspecified nonlinear AS where we impose  $\kappa^L = (\kappa^L)' = 0$ . For small  $p$  it is upward sloping. For large  $p$  though the AS can be downward sloping. However, this possibility appears to be remote in practice. Most empirical parameterizations have the property that  $p < 0.95$  and that  $\beta d^L$  is only marginally above one so that  $1 > p\beta d^L$  and the AS schedule is upward sloping (see e.g. [Christiano, Eichenbaum, and Rebelo \(2011\)](#); [Eggertsson \(2011\)](#); and [Woodford \(2011\)](#)).

**Lemma 2** *If  $1 > p\beta d^L$  then the misspecified AS nonlinear schedule is upward sloping.*

Recognizing the resource costs of price adjustment introduces a new term into the numerator of (24) and this term is negative. It is thus possible that the AS schedule is downward sloping for standard parameterizations of the NK model.

### 3.2 Numerical results

The above results provide weak regularity conditions under which the AD and AS schedules are both upward sloping. This is the case for both the log-linearized solution and the nonlinear solution to the misspecified model where  $\kappa^L = (\kappa^L)' = 0$ . We do not know however, the relative slope of the two schedules and we do not have information on the slopes of the AD and AS schedules for the “true” nonlinear model except in some special cases.

We now turn to report numerical results that are designed to produce a more complete characterization of the equilibrium for the “true” nonlinear model and also to pin down the relative slopes of the AD and AS schedules in the misspecified nonlinear model. An advantage of our setup is that we can compute the exact equilibrium for a given parameterization up to the accuracy of the computer. There is no need to use perturbation or projection methods to compute a solution.

One of the main questions we face though is deciding what is a reasonable parameterization of the model? We start by considering a parameterization of our model that yields an identical log-linearized decision rule and thus identical impulse responses to a shock in  $d$  to those reported in Eggertsson (2011). For the convenience of the reader we report his parameterization using our notational conventions in Table 1.<sup>5</sup>

Although the log-linear dynamics of our model and Eggertsson (2011) model are identical, we wish to emphasize that there are some differences in the nonlinear model that he considers.

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<sup>5</sup>We explain in a Technical Appendix available on request from the authors the derivations of the formula that link the Calvo parameter in Eggertsson (2011) to  $\gamma$  in our model. The main thing to keep in mind is that we hold fixed the labor supply elasticity at the value chosen by Eggertsson (2011) and only adjust the parameter that governs the resource costs of price adjustment.

Table 1: Parameterization

| <i>Symbol</i> | Value   | Description                                      |
|---------------|---------|--|
| $\beta$       | 0.997   | Discount factor                                  |
| $\sigma$      | 1.1599  | Consumption curvature                            |
| $\nu$         | 1.5692  | Leisure curvature                                |
| $p$           | .9030   | Probability of a low state in the next period    |
| $\theta$      | 12.7721 | Elasticity of substitution of intermediate goods |
| $\alpha$      | 0.7747  | Calvo Parameter                                  |
| $\gamma$      | 3742.9  | Implied price adjustment cost parameter          |
| $\tau_w$      | 0.2     | Labour tax rate                                  |
| $d^L$         | 1.015   | Preference discount factor shock                 |
| $\eta$        | 0.2     | Government purchase share of output              |

[Eggertsson \(2011\)](#) assumes Calvo pricing and that each type of intermediate good is produced with a distinct type of labor. He also assumes that the tax on labor is a payroll tax. The derivations and intuition are considerably more transparent using our setup with Rotemberg price adjustment. Moreover, as emphasized by [Christiano \(2011\)](#), the effects of a change in the labor tax and payroll tax have the same effects when wages are flexible as we assume here. Still the reader should keep in mind that when we refer to results for the “true” model it is for the model described above in [Section 2](#) and not the model of [Eggertsson \(2011\)](#).

Consider now a shock to the preference discount factor of the size posited by [Eggertsson \(2011\)](#) ( $d^L = 1.015$ ). Column 1 in [Table 2](#) reproduces [Eggertsson \(2011\)](#) results for the log-linearized system. Both the AS and AD schedules are upward sloping and the AD schedule cuts the AS schedule from below (see Panel A of [Figure 1](#)). Column 2 reports the slopes of the nonlinear misspecified AD and AS that arise when one imposes  $\kappa^L = (\kappa^L)' = 0$ . We already know from [Lemma 1](#) that the misspecified nonlinear model delivers an upward sloping AD schedule too. This parameterization satisfies the condition of [Lemma 2](#) so the AS schedule is also upward sloping. Most importantly the AS schedule has a smaller slope than the AD schedule and thus also corresponds to Panel B in [Figure 1](#). We will see below that this configuration of the AD and AS schedules implies that labor input falls in response to a tax cut.

Figure 1  
Configurations of aggregate demand and aggregate supply schedules that arise when nominal rate is zero.

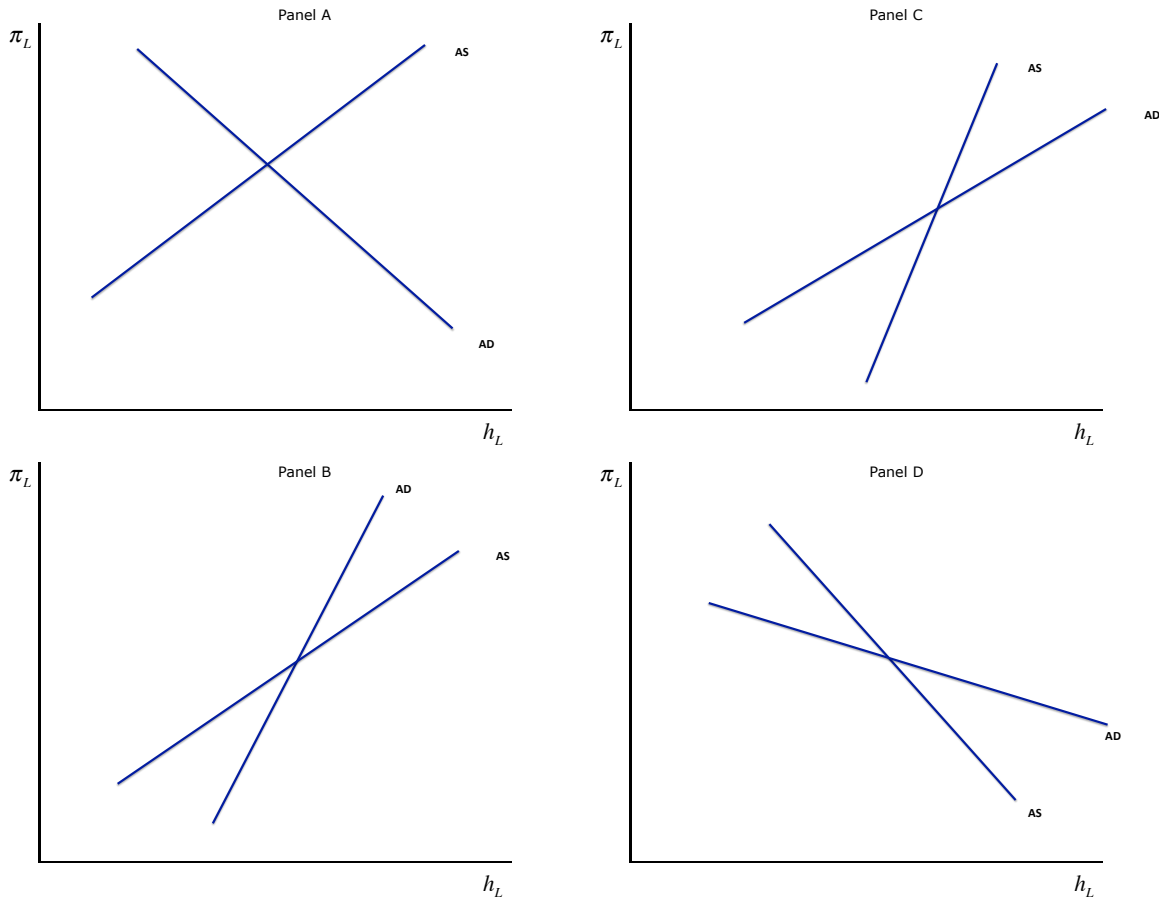


Table 2

Properties of low (zero interest rate) state using Eggertsson (2011) parameterization of the model

| Solution Procedure         | log-linearized | nonlinear (kappa=0) | nonlinear |
|----------------------------|----------------|---------------------|-----------|
| Percentage change in hours | -29.92%        | -34.70%             | -2.20%    |
| Inflation rate             | -9.92%         | -8.81%              | -3.10%    |
| Slope of AS schedule       | 0.12           | 0.06                | -0.19     |
| Slope of AD schedule       | 0.17           | 0.11                | -0.03     |

Note that also that there is evidence of some small approximation errors that are unrelated to the treatment of the resource costs of price adjustment. A comparison of the first and second columns of Table 2 indicates that the response of output is larger using the nonlinear model and the inflation response is smaller.

From the discussion following Propositions 1 and 2 we know that when  $\kappa^L = (\kappa^L)' = 0$ , the slopes will vary with the size of the shock to  $d$  but that the qualitative properties of the model will always resemble those in Panel B of Figure 1. We also know from Eggertsson (2011) that the same properties are true for the log-linearized system.

However, we don't know the properties of the true nonlinear model for large  $p$ . We saw above that it was only possible to establish analytical results for the "correct" model when  $p$  is sufficiently small. The value of  $p = 0.9030$  in the Eggertsson (2011) parameterization is very far from zero.

Column 3 reports results for the nonlinear model that recognizes the resource costs of price adjustment. The results are quite different from the other two specifications. Both the AD and the AS schedules are downward sloping! Moreover, the AS schedule cuts the AD schedule from above as shown in Panel C of Figure 1.

One way to understand the distinction between the "true" solution and the various approximate solutions is to consider Figures 2-4. These figures allow us to isolate the effect of the resource costs of price adjustment. Figure 2 consists of three curves in the  $c$  and  $h$  space. The iso-marginal utility of consumption curve, the aggregate resource constraint conditional on  $\kappa = 0$  and the household intratemporal first order condition (FONC) conditional on marginal cost being equal to one. The steady-state equilibrium occurs at point A.

Suppose now that we have a positive shock to  $d$ . This experiment is reported in Figure 3. This shock induces shifts in two of the schedules. First, households become more patient and the marginal utility of current consumption increases  $u_c > \underline{u}_c$ . The second thing that happens is that marginal cost falls below one which shifts the intratemporal FONC to the left. The result is that both hours and consumption fall from point A to point B.

Figure 2: Model in the steady-state

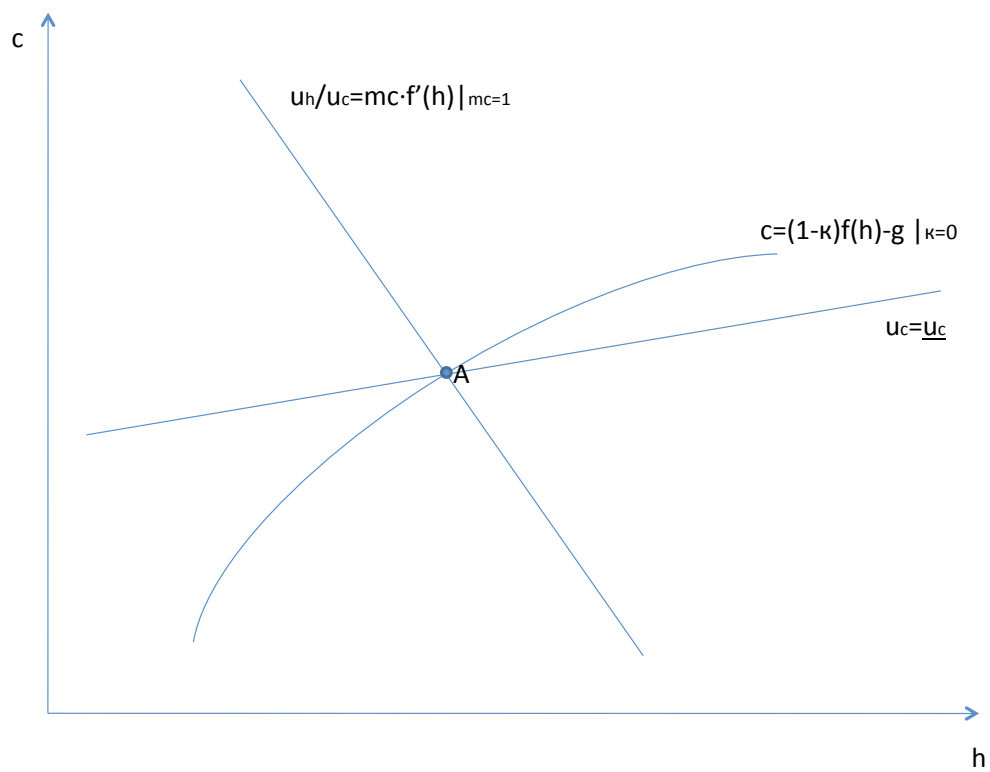
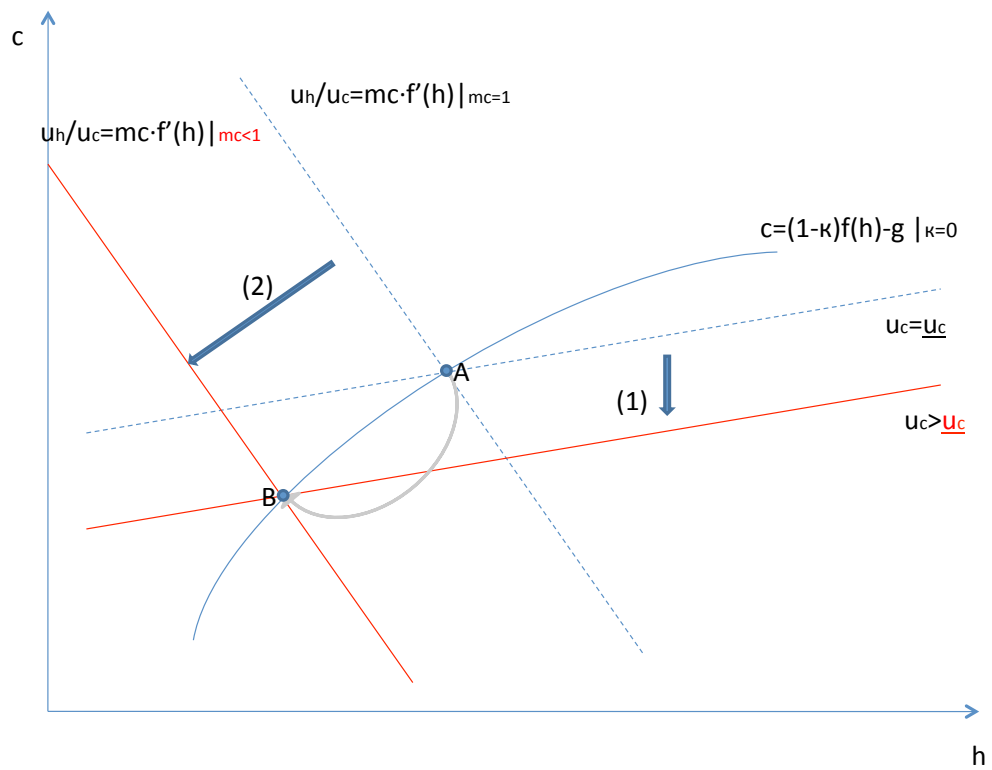
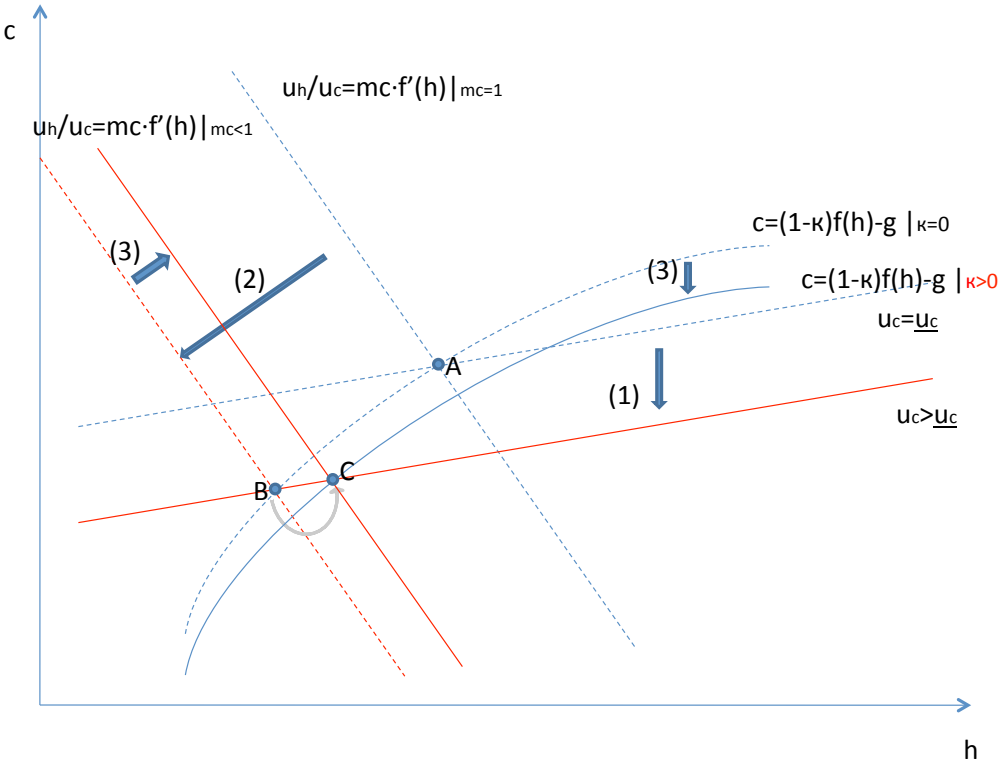


Figure 3: Model following a  $d$  shock with  $\kappa = 0$



Consider next what happens when the resource costs of price adjustment are recognized (see Figure 4). There are two effects. First, the aggregate resource constraint shifts down. Less of gross production is available for consumption. Second, the intratemporal FONC schedule shifts right from point B to C. From this we see that recognizing the resource costs of price adjustment dampens the response of marginal cost and thereby mitigates the declines in consumption and hours.

Figure 4: Model following a  $d$  shock with  $\kappa > 0$



The “correct” solution has some other interesting properties. The sign of the slopes of the AS

and the AD schedules varies with the size of the  $d$  shock and there are sometimes also two bonafide zero bound equilibria with distinct configurations of the AS and AD schedules.

Table 3 reports how the properties of the model change with with size of the shock to  $d$  using the Eggertsson (2011) calibration of the “true” model. Table 3 has two panels one for each of the two solutions to equations (17) and (18).<sup>6</sup>

Table 3  
Zero interest rate equilibria for alternative sized shocks to preferences using Calvo parameter of 0.7747.

| <i>First Equilibrium</i>              |        |        |        |        |       |         |       |       |       |
|---------------------------------------|--------|--------|--------|--------|-------|---------|-------|-------|-------|
| Annualized increase in d (percentage) | 10.587 | 9.612  | 9.127  | 7.401  | 5.514 | 5.468   | 2.973 | 2.927 | 2.454 |
| Percentage change in hours            | 1.54   | 0.14   | -0.47  | -2.20  | -3.15 | -3.15   | -1.78 | -1.69 | -0.14 |
| Annualized inflation rate             | -4.14  | -3.85  | -3.69  | -3.09  | -2.31 | -2.29   | -0.74 | -0.69 | -0.05 |
| Percentage change in GDP              | -21.67 | -19.49 | -18.39 | -14.37 | -9.73 | -9.61   | -2.42 | -2.26 | -0.14 |
| Slope of AS schedule                  | -0.06  | -0.08  | -0.09  | -0.19  | -7.87 | 1294.19 | 0.13  | 0.12  | 0.09  |
| Slope of AD schedule                  | -0.02  | -0.02  | -0.02  | -0.03  | -0.05 | -0.05   | -2.50 | 9.84  | 0.14  |
| <i>Second Equilibrium</i>             |        |        |        |        |       |         |       |       |       |
| Annualized increase in d (percentage) | 10.587 | 9.612  | 9.127  | 7.401  | 5.514 | 5.468   | 2.973 | 2.927 | 2.454 |
| Percentage change in hours            | 20.36  | 18.93  | NA     | NA     | NA    | NA      | NA    | NA    | NA    |
| Annualized inflation rate             | 4.15   | 3.95   | NA     | NA     | NA    | NA      | NA    | NA    | NA    |
| Percentage change in GDP              | -1.33  | -0.53  | NA     | NA     | NA    | NA      | NA    | NA    | NA    |
| Slope of AS schedule                  | 0.02   | 0.03   | NA     | NA     | NA    | NA      | NA    | NA    | NA    |
| Slope of AD schedule                  | 0.01   | 0.01   | NA     | NA     | NA    | NA      | NA    | NA    | NA    |

Consider the results reported in the upper panel of Table 3 under the heading *First Equilibrium*. This solution is a bonafide zero bound equilibrium for the entire range of shocks to  $d$  that we consider. Notice that each of the four configurations of the AD and AS schedule displayed in Figure 1 occur. For  $d$  shocks that range from 5.5% to 9.1% the AD and AS schedules are both downward sloping and both hours and GDP fall. For shocks that are larger than 5.5% both schedules are downward sloping but the response of hours and GDP is different. GDP falls but labor input actually increases relative to its steady-state level!

The distinction between the response of GDP and hours arises because GDP is net of the resource costs of price adjustment. For values of  $d \geq 9.6\%$  the resource costs are so large that the sign response of these two variables is different. We will say more about this distinction below.

For  $2.97\% < d < 5.5\%$  we see that the AD and AS schedules both have their conventional slopes. The AD schedule is downward sloping and the AS schedule is upward sloping. Indeed it

<sup>6</sup>We limit attention to real valued solutions.

is only for very small shocks to  $d$  of from 2.45% to 2.93% that we observe a situation like Panel B in Figure 2 with the AD and AS schedules both upward sloping and the AD schedule cutting the AS schedule from below.

To get some intuition for these results observe from equation (24), it follows that as the size of  $d^L$  is reduced from the value considered by Eggertsson (2011) the slope of the AS schedule will eventually turn positive. From (22) we can also see that a smaller value of  $d^L$  reduces the size of the AD slope in absolute value. How, far can it be reduced though?

On the one hand it is well known that log-linearized systems work well for small shocks. This implies that for small enough shocks the nonlinear and log-linear solutions should be close. However, what is less clear is whether this approximation can still work for shocks that are large enough to get the nominal interest rate to fall to zero. The results in Table 3 indicate that there is a small interval of shocks to  $d$  that have the requisite properties: they are large enough to get the nominal rate to hit zero but they are also small enough such that the nonlinear solution and the log-linear solutions are close enough such that the slope of the AD and AS schedules are both positive and the AD schedule cuts the AS schedule from below.

The results we derived above imply that hours always falls if the resource costs of price adjustment are abstracted from. Using Figure 4 this implies that the shift down in the iso-utility schedule is more than offset by a leftward shift in the intratemporal FONC. However, in Table 3 we see that hours can actually lie above their steady-state level after an increase in  $d$ . This can also be understood by considering Figure 4. Once the resource costs of price adjustment are recognized big shocks to  $d$  can induce an adjustment so large that point C lies to the right of point A and hours actually increase.

Consider next the lower panel of Table 3. We find that (17) and (18) generally have two solutions. In many cases though the second solution is not a bonafide zero bound equilibrium because the implied value of the nominal interest rate is positive. These columns are labeled NA. In the cases where the second solution is a bonafide zero bound equilibrium, it has the property that the AD and AS schedules are both upward sloping however, the AD schedule always cuts

that AS schedule from above (see Panel C in Figure 1). This distinction is important because a tax cut only has a contractionary effect on labor input if the AD schedule cuts the AS schedule from below. Its properties though are rather unorthodox. Hours and inflation both lie above their steady-state levels. GDP is below its steady-state level though when the (positive) shock to  $d$  is sufficiently large.

Table 4

Zero interest rate equilibria for alternative sized shocks to preferences using Calvo parameter of 0.634.

| <i>First Equilibrium</i>                |        |        |       |       |       |       |       |       |        |       |       |
|---|--------|--------|-------|-------|-------|-------|-------|-------|--------|-------|-------|
| Annualized increase in $d$ (percentage) | 10.46  | 4.860  | 2.552 | 2.470 | 2.307 | 2.184 | 1.940 | 1.857 | 1.852  | 1.848 | 1.845 |
| Percentage change in hours              | 21.32  | 4.97   | 0.04  | -0.13 | -0.44 | -0.60 | -1.01 | -1.08 | -1.08  | -1.08 | -1.07 |
| Annualized inflation rate               | -10.02 | -6.95  | -4.48 | -4.32 | -3.97 | -3.77 | -2.91 | -2.40 | -2.33  | -2.24 | -2.11 |
| Percentage change in GDP                | -30.11 | -15.96 | -7.96 | -7.55 | -6.66 | -6.17 | -4.27 | -3.28 | -3.16  | -2.98 | -2.76 |
| Slope of AS schedule                    | -0.03  | -0.09  | -0.25 | -0.28 | -0.34 | -0.40 | -1.05 | -9.38 | 774.23 | 6.38  | 2.78  |
| Slope of AD schedule                    | -0.01  | -0.04  | -0.11 | -0.12 | -0.14 | -0.16 | -0.38 | -1.38 | -2.10  | -8.44 | 2.84  |
| <i>Second Equilibrium</i>               |        |        |       |       |       |       |       |       |        |       |       |
| Annualized increase in $d$ (percentage) | 10.46  | 4.860  | 2.552 | 2.470 | 2.307 | 2.184 | 1.940 | 1.857 | 1.852  | 1.848 | 1.845 |
| Percentage change in hours              | NA     | NA     | NA    | NA    | -0.23 | -0.38 | -0.86 | -1.02 | -1.04  | -1.05 | -1.07 |
| Annualized inflation rate               | NA     | NA     | NA    | NA    | -0.27 | -0.46 | -1.30 | -1.81 | -1.87  | -1.97 | -2.10 |
| Percentage change in GDP                | NA     | NA     | NA    | NA    | -0.25 | -0.46 | -1.50 | -2.26 | -2.37  | -2.53 | -2.74 |
| Slope of AS schedule                    | NA     | NA     | NA    | NA    | 0.32  | 0.35  | 0.63  | 1.21  | 1.38   | 1.74  | 2.68  |
| Slope of AD schedule                    | NA     | NA     | NA    | NA    | 0.14  | 0.16  | 0.31  | 0.70  | 0.83   | 1.18  | 2.62  |

Finally, let's consider which of the shocks to  $d$  comes closest to reproducing the outcomes from the Great Depression.

Eggertsson (2011) parameterized his model to reproduce two targets from the Great Depression: a 10% decline in the inflation rate a 30% decline in output. None of the specifications reported in Table 3 reproduce these outcomes. However, it is interesting that the specification that comes closest is the *First Equilibrium* with the largest shock to  $d$ . That specification produces a 4% decline in inflation and a 22% decline in GDP. There are two other things to note about this parameterization. Hours is above its steady-state value and there are two equilibria.

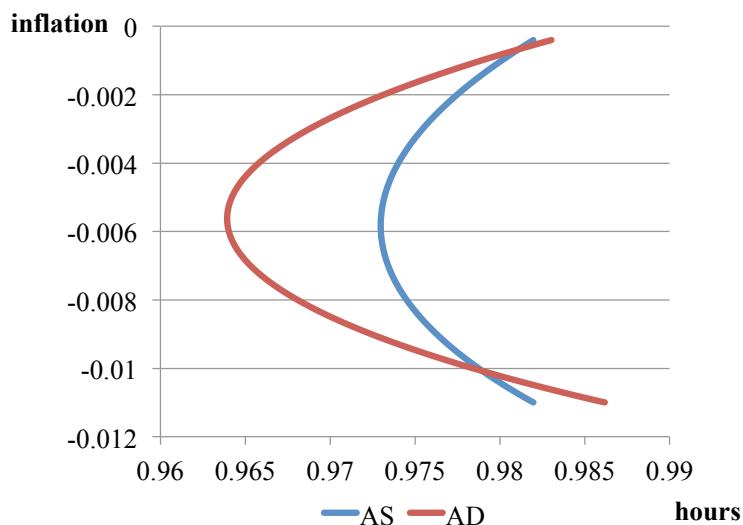
So far we have limited attention to the same calibration used by Eggertsson (2011). That calibration however does not reproduce Eggertsson's targets for the great depression of a 30% GDP decline and a 10% decline in the inflation rate even if we consider larger shocks to  $d$ . We now consider what happens if we recalibrate the "correct" nonlinear model to reproduce these declines. We adjust the size of the shock to  $d$  and the size of  $\gamma$  to match these two targets. Table

4 indicates that a shock to  $d$  that increases the preference discount rate by 10.4% ( $d_L = 1.022$ ) in conjunction with a value of  $\gamma$  that implies a Calvo parameter of 0.634 successfully reproduces these two targets. For this parameterization of the model there is a unique equilibrium and the AD and AS schedules are both downward sloping with the AS schedule steeper than the AD schedule. Observe that there continues to be a qualitative difference between the hours response and the GDP response. GDP is down by 30% but hours are 21% above their steady-state level! Any difference between the response of GDP and hours is attributable to the response of the resource costs and it can thus be seen that the resource costs of price adjustment play a central role for the dynamics of the model. Table 4 also reports results for smaller shocks to  $d$ . For this parameterization of the model the region where the AD schedule is upward sloping and cuts the AS schedule from below is very small. The interval of  $d$  shocks in which the AD and AS schedules have their conventional slopes is also tiny. The remaining values of  $d$  have the property that both the AD and AS schedules are downward sloping.

The second equilibrium once again has the property that the AD and AS schedules are upward sloping but that the AD schedule cuts the AS schedule from above. For the previous parameterization of the model this second equilibrium was not particularly interesting because it had the property that inflation and hours were both well above their steady-state values. Here the two equilibria are remarkably similar. Consider for instance the case where the shock to  $d$  is 2.307%. For this sized shock both equilibria are associated with declines in hours, inflation and GDP. To illustrate how the two equilibria arise we plot the AD and AS schedules for this parameterization of the model in Figure 5. Observe that both schedules are convex. The main distinction between the two equilibria is that the slopes of the AD and AS schedules are always different.

The U.S. economy has changed considerably since the Great Depression and it is interesting to understand the properties of the model if one sets the size of the shock to  $d$  and  $\gamma$  to reproduce events from a more recent event. [Christiano, Eichenbaum, and Rebelo \(2011\)](#) parameterize their model to reproduce declines in inflation and output that are consistent with outcomes during the

Figure 5: AD and AS schedules from a parameterized version of the model with  $\gamma = 0.6364$ ,  $d_L = 1.0027$ .



recent financial crisis in the U.S. They target a one percent decline in the inflation rate and a seven percent decline in output. When we recalibrate the “correct” model to reproduce these outcomes by altering  $\gamma$  and  $d$  while holding fixed the other parameters, the resulting value of the Calvo parameter is 0.836 and the value of  $d_L = 1.0096$ . It turns out that there is a unique equilibrium and the sign of the AD schedule is negative and the sign of the AS schedule is positive.

We turn now to consider the implications of these findings for the response of the economy to a labor tax cut.

#### 4 The response of labor input to a labor tax cut in a liquidity trap.

With a characterization of the slopes of the AS and AD schedules in hand we can now analyze the response of labor input to a tax cut. The results that follow are quite intuitive. Our simulation results produced the four configurations of the AS and AD schedules reported in Figure 1. From

(17) and (18) we know that a labor tax cut acts to shift the AS schedule outward to the right along a stable AD schedule. Using informal graphical arguments one would expect that a tax cut would increase hours worked and lower inflation in Panel A which corresponds to an orthodox situation. If the schedules are configured in Panel B a tax cut acts to lower labor input and inflation as shown in Panel B. This is the finding highlighted by Eggertsson (2011). It should be kept in mind that to get this result it is important that the AD schedule be not only positive but also be steeper than the AS schedule. When Panels C or D are relevant a labor tax cut also has orthodox properties: hours increase and inflation falls.

We next provide some formal results that demonstrate that this intuition is correct.

In analyzing the effect of a change in the labor tax,  $\tau_{w,t}$ , on prices and allocations in state  $L$  we consider infinitesimal cuts in the labor tax rate  $\tau_{w,t}$  of a constant and fixed size that last as long as the nominal rate is zero. In particular,  $\tau_w^L < \tau_w$  for all periods in which  $d_t = d^L$ .

To derive the responses start by applying the chain rule to (17) and (18) to get

$$AS_{h^L} Dh^L + AS_{\pi^L} D\pi^L + AS_{\tau_w^L} D\tau_w^L = 0 \quad (26)$$

$$AD_{h^L} Dh^L + AD_{\pi^L} D\pi^L = 0 \quad (27)$$

We then solve the total differential of the AD schedule for  $D\pi^L$  and substitute it into the total differential of the AS schedule to get:

$$\left( AS_{h^L} - AS_{\pi^L} \frac{AD_{h^L}}{AD_{\pi^L}} \right) Dh^L + AS_{\tau_w^L} D\tau_w^L = 0 \quad (28)$$

or

$$\frac{Dh^L}{D\tau_w^L} = - \frac{AS_{\tau_w^L}}{AS_{h^L} - AS_{\pi^L} \frac{AD_{h^L}}{AD_{\pi^L}}} \quad (29)$$

where the various derivatives are as follows

$$AS_{h^L} = \frac{\theta(\sigma + \nu)(1 - \kappa^L - \eta)^\sigma (h^L)^{\sigma + \nu - 1}}{(1 - \tau_w^L)} \quad (30)$$

$$AS_{\pi^L} = -\frac{\theta\sigma(1 - \kappa^L - \eta)^{\sigma - 1}(\kappa^L)'(h^L)^{\sigma + \nu}}{(1 - \tau_w^L)} + \gamma(p\beta d^L - 1)(1 + 2\pi^L) \quad (31)$$

$$AS_{\tau_w^L} = \frac{\theta(1 - \kappa^L - \eta)^\sigma (h^L)^{\sigma + \nu}}{(1 - \tau_w^L)^2} \quad (32)$$

$$AD_{\pi^L} = -\frac{p\beta d^L}{(1 + \pi^L)^2} + \frac{(p - 1)\sigma(h^L)^\sigma(1 - \kappa^L - \eta)^{\sigma - 1}(\kappa^L)'}{(1 - \eta)^\sigma h^\sigma} \quad (33)$$

$$AD_{h^L} = \frac{\sigma(1 - p)(1 - \kappa^L - \eta)^\sigma (h^L)^{\sigma - 1}}{(1 - \eta)^\sigma h^\sigma} \quad (34)$$

The following three Lemmas establish a formal link between the sign and magnitude of the slopes of the AD and AS schedules and the response of hours and inflation to a labor tax cut.

### Lemma 3

Suppose  $0 < p < 1$ , the AS schedule is upward sloping, the AD schedule is also upward sloping and cuts the AS schedule from below as in Panel B of Figure 1, then hours fall when the labor tax is cut ( $\frac{Dh^L}{D\tau_w^L} > 0$ ) and the inflation rate also falls.

### Proof

Consider again

$$\begin{aligned} \frac{Dh^L}{D\tau_w^L} &= -\frac{AS_{\tau_w^L}}{AS_{h^L} - AS_{\pi^L} \frac{AD_{h^L}}{AD_{\pi^L}}} \\ &= \frac{-AS_{\tau_w^L}}{AS_{\pi^L}} \\ &= \frac{\frac{AS_{h^L}}{AS_{\pi^L}} - \frac{AD_{h^L}}{AD_{\pi^L}}}{\frac{-AS_{\tau_w^L}}{AS_{\pi^L}}} \\ &= \frac{-slope(AS) + slope(AD)}{-slope(AS) + slope(AD)} \end{aligned}$$

The assumption that the AS schedule is positive implies that  $AS_{\pi^L} < 0$  and inspection of (32)

reveals that  $AS\tau_w^L > 0$ . It follows that the numerator is positive. Thus, it is sufficient to show that the denominator is also positive. However, this follows from the fact that the slope of the AD schedule is steeper than the slope of the AS schedule. Having established that hours falls in response to a tax cut, the assumption that the AS schedule is upward sloping means that (24) is positive and that the inflation rate falls.  $\square$

Using the same logic it is straightforward to establish that the model delivers orthodox responses to a labor tax cut when the AD schedule is downward sloping and the AS schedule is upward sloping as in Panel A of Figure 2.

**Lemma 4.**

*Suppose that the AS schedule is upward sloping and the AD schedule is downward sloping as in Panel A of Figure 1, then hours increase when the labor tax is cut:  $\left(\frac{Dh^L}{D\tau_w^L} < 0\right)$  and the inflation rate falls.*

When both the AD and the AS schedules are upward sloping but the AS schedule cuts the AD schedule from below as in Panel C of Figure 1 hours increases in response to a tax cut. In contrast to the other cases though the inflation rate also increases.

**Lemma 5.**

*Suppose that the AS schedule is upward sloping and the AD schedule is upward sloping but cuts the AS schedule from above as in Panel C of Figure 1, then hours increase when the labor tax is cut:  $\left(\frac{Dh^L}{D\tau_w^L} < 0\right)$  the inflation rate rate also increases.*

Finally, the model produces orthodox responses of hours and the inflation rate to a labor tax cut if the AS schedule is downward sloping and steeper than the AD schedule as shown in Panel D of Figure 1.

**Lemma 6.**

Suppose that the AD schedule is downward sloping and the AS schedule is also downward sloping with  $\text{slope}(|AS|) > \text{slope}(|AD|)$  as in Panel D of Figure 1 then hours increase when the labor tax is cut:  $\left(\frac{Dh^L}{D\tau_w^L} < 0\right)$  and the inflation rate falls.

Using Lemmas 3- 6 we can establish the following difference between the nonlinear solution of the true model and the nonlinear solution of the model that abstracts from the resource costs of price adjustment and the log-linearized solution when  $p = 0$ .

### Proposition 3

1. When  $p = 0$  a tax cut increases labor input in the model. However, the log-linear and nonlinear solutions that abstract from the resource costs of price adjustment imply instead that labor input does not change.
2. For  $p > 0$  but sufficiently small, a tax cut increases labor input in the model. The log-linear and nonlinear solutions that abstract from the resource costs of price adjustment incorrectly imply that labor input should fall.

We now illustrate using numerical methods that these results are also relevant when  $p$  is large and the model is parameterized in a conventional way.

Table 3 reports results for small perturbations in the labor tax for the same specifications considered in Table 2. Recall from Table 2 that the log-linearized system reported in column 1 has the property that the AD and AS schedules are both upward sloping and that the AD schedule cuts the AS schedule from below. From Lemma 3 we know that this implies that a tax cut lowers hours. As in [Eggertsson \(2011\)](#) the size of the multiplier is 1. Column 2 reports the labor tax multiplier for the misspecified nonlinear system. Lemma 3 applies here too and hours fall in response to a tax cut. Observe also that the labor tax multiplier is smaller. It has fallen from 1 using the log-linearized solution to 0.56. Finally, column 3 reports results using the correct model consistent resource constraint. From Lemma 5 we know that the multiplier is negative (hours increase when the labor tax is cut). We can see that the difference is quite substantial. The labor tax multiplier

is now -0.56.

Lemmas 3-6 can also be used to directly ascertain the hours response to tax cuts in the other specifications we considered above. Consider, for instance, the results in Table 3. Applying Lemmas 3-6 to the *first equilibrium* implies that a labor tax cut raises labor in the “true” model for all shocks except the small interval between 2.927 and 2.454. It is worth noting that the 2.454%  $d$  shock scenario is the smallest sized shock to  $d$  that is consistent with a zero nominal interest rate. In the *second equilibrium* hours always increases in response to a tax cut.

Consider next the recalibrated model results reported in Table 4. Recall that the left most column of that table reports results that reproduce the Great Depression. For that parameterization of the model Lemma 4 implies that labor input increases in response to a tax cut. More, generally hours also increase in response to a tax cut for all shocks to  $d$  that are 1.8% or larger.

Using Lemma 4 we also know that hours increases in response to a labor tax cut when we parameterize the model to reproduce output and inflation responses from the recent financial crisis.

Overall, these results imply that labor input generally increases in response to a labor tax cut. In particular, hours increase using the [Eggertsson \(2011\)](#) parameterization of our model. They also increase when we recalibrate our true model to reproduce the Great Depression. They also increase with we use the [Christiano, Eichenbaum, and Rebelo \(2011\)](#) targets from the recent financial crisis. More generally labor input only falls for shocks that lie in a very small neighborhood of the point where the nominal interest rate falls to zero.

These findings are also consistent with previous research by [Braun and Körber \(2011\)](#). They also find that the hours increase in response to a tax cut in a NK model with capital accumulation that is calibrated to reproduce Japan’s experience with zero interest rates. They use a global solution method to solve their model.

Table 3 also reports results for the response of GDP. When the resource costs of price adjustment are ignored there is no distinction between hours and GDP. Recall though that in the true

Table 3  
Labor tax hours and GDP multipliers

| Solution Procedure                     | log-linearized | nonlinear ( $\kappa^l=0$ ) | nonlinear |
|--|----------------|----------------------------|-----------|
| Labor Tax Hours Multiplier             | 1.02           | 0.57                       | -0.56     |
| Labor Tax GDP Multiplier               | 1.02           | 0.57                       | 0.02      |
| $\Delta \ln(1-\kappa^l)/\Delta \tau_w$ | 0              | 0                          | 0.58      |

model GDP and hours are related by (13). Interestingly, the results in Column 3 show that the sign of the responses of the two variables is different. Hours increases with a tax cut, but GDP falls by a small amount. To see how this can happen observe that the GDP labor tax multiplier can be expressed as:

$$\frac{\Delta \ln(gdp^L)}{\Delta \tau_w^L} = \frac{\Delta \ln(1 - \kappa^L)}{\Delta \tau_w^L} + \frac{\Delta \ln(h^L)}{\Delta \tau_w^L} \quad (35)$$

This decomposition shows that GDP can falls with a labor tax cut if the savings in resource costs associated with a higher price level are sufficiently large. The final row of Table 3 reports the value of  $\frac{\Delta \ln(1-\kappa^L)}{\Delta \tau_w^L}$ . These savings are quite substantial and indeed large enough to overwhelm the negative response of gross output (hours). We see from this that a tax cut lowers GDP in the nonlinear model. However, the economic mechanism underlying this increase is entirely difference. GDP falls because the savings in resource costs associated with a higher price level are very large.

## 5 Robustness

Here we briefly report some of the checks for robustness that we have performed. First, we have investigated the properties of the model for larger values of  $\nu$ . A larger value of  $\nu$  reduces the value of  $\gamma$  and the Calvo parameter needed reproduce the Great Depression targets. If  $\nu$  is increased to 2 there is no interval of  $d$  for which a tax cut reduces labor input. We have also solved an “exact” perfect foresight version of the Calvo model with a homogenous labor market. We calibrate the

model to reproduce alternatively the Great Depression targets or the Financial Crisis targets of [Christiano, Eichenbaum, and Rebelo \(2011\)](#). Under either parameterization, the model has the property that a tax cut increases labor input.

## 6 Conclusion

A large body of recent research has analyzed the zero bound by taking a short cut. That short cut is to log-linearize all equilibrium conditions except for the monetary policy rule around a steady-state with a stable price level. This paper has illustrated that this common practice can result in mistaken inference. In the example considered here the log-linearized solution predicts unambiguously that a labor tax cut will lower labor input. We have shown that the exact solution has the opposite property for a large and empirically relevant range of model parameters.

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